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MATHEMATICS

On a Linear Differential Inequality of Parabolic Type

by

W. MLAK

Presented by T. WAŻEWSKI on August 28, 1959

In this paper we are concerned with the differential inequality $\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2}$. It is a classical result that this inequality together with some boundary properties of u implies that $u \leqslant 0$. It is supposed that the differential inequality holds everywhere in a suitable open plane set. We are interested in finding conditions on the function u which guarantee that $u \leqslant 0$ whenever the inequality $\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2}$ holds almost everywhere. Throughout the present paper R denotes the rectangle defined by the inequalities $0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T$. Let $G(x, t;\ \xi, \tau)$ be the Green's function of the first boundary value problem for the equation $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$ considered for R. The following properties of $G(x, t;\ \xi, \tau)$ will be used (see for instance [2])

(1) $G(x, t;\ \xi, \tau) \geqslant 0$ for $(x, t) \in R, 0 < \tau < t \leqslant T$ and $0 \leqslant \xi \leqslant 1$,

(2) $G(x, t;\ 0, \tau) = G(x, t;\ 1, \tau) = 0$ for $(x, t) \in R, 0 < \tau < t \leqslant T$,

(3) $$\frac{\partial^2 G(x, t;\ \xi, \tau)}{\partial \xi^2} + \frac{\partial G(x, t;\ \xi, \tau)}{\partial \tau} = 0 \quad \text{for} \quad (x, t) \in R$$

and

$$0 < \tau < t \leqslant T, 0 \leqslant \xi \leqslant 1 .$$

(4) $\lim\limits_{t \to s+} \int_0^1 G(x, t;\ \xi, s) \varphi(\xi, s) d\xi = \varphi(x, s)$ whenever $\varphi(x, s)$ is continuous in R.

For the sake of simplicity we assume in the following that the function $u(x, t)$ is continuous in R. We shall make use of the following assumptions: *)

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*) Similar regularity properties of solutions of parabolic equations have been discussed by B. Pini in [3].

(5) For almost all $x \in [0,1]$ the function $u(x,t)$ is absolutely continuous with respect to t $(t \in [0,T])$.

(6) For almost all $t \in [0,T]$ the derivative $\dfrac{\partial u(x,t)}{\partial x}$ exists for all $x \in [0,1]$ and is absolutely continuous with regard to x $(x \in [0,1])$.

(7) The derivative $\dfrac{\partial^2 u}{\partial x^2}$ is summable in R, i. e. $\iint\limits_R \left|\dfrac{\partial^2 u}{\partial x^2}\right| dx\,dt < +\infty$.

We shall prove the following theorem:

THEOREM. *Let $u(x,t)$ satisfy the conditions (5)-(7). Suppose that the following conditions hold:*

(8) $u(x,0) \leqslant 0$ *for* $x \in [0,1]$ *and* $u(1,t) = u(0,t) = 0$ *for* $t \in [0,T]$.

It is assumed that the inequality

(9)
$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2}$$

holds almost everywhere in R. Then $u(x,t) \leqslant 0$ for $(x,t) \in R$.

Proof *): Let us define the function v by means of the following formula:

$$v(x,t;\ s) = \int_0^1 G(x,t;\ \xi,s)\,u(\xi,s)\,d\xi\,;\ s < t\,.$$

Write now

(10)
$$\Delta_h = v(x,t;\ s,h) - v(x,t;\ s)\,; \quad h > 0 \quad s+h < t\,.$$

It follows from the regularity properties of G and from (5) that the function $G(x,t;\ \xi,\tau)\,u(\xi,\tau)$ is absolutely continuous with regard to τ for almost all ξ. We derive therefore that

(11)
$$G(x,t;\ \xi,s+h)\,u(\xi,s+h) - G(x,t;\ \xi,s)\,u(\xi,s) = \int_s^{s+h} \frac{\partial}{\partial\tau}\big(G(x,t;\ \xi,\tau)\,u(\xi,\tau)\big)\,d\tau$$

for almost all $\xi \in [0,1]$.

By (10) and (11) we get

(12)
$$\Delta_h = \int_0^1 \left\{\int_s^{s+h} \left[\frac{\partial G(x,t;\ \xi,\tau)}{\partial\tau}\,u(\xi,\tau) + G(x,t;\ \xi,\tau)\,\frac{\partial u(\xi,\tau)}{\partial\tau}\right] d\tau\right\} d\xi\,.$$

On the other hand, (1) and (9) imply that

$$\int_s^{s+h} G(x,t;\ \xi,\tau)\,\frac{\partial u(\xi,\tau)}{\partial\tau}\,d\tau \leqslant \int_s^{s+h} G(x,t;\ \xi,\tau)\,\frac{\partial^2 u(\xi,\tau)}{\partial\xi^2}\,d\tau$$

*) For the properties of Lebesgue integral we refer to [1].

for almost all $\xi \in [0, 1]$. We obtain therefore

$$(13)\quad \int_0^1 \left[\int_s^{s+h} G(x, t;\ \xi, \tau)\frac{\partial u(\xi, \tau)}{\partial \tau} d\tau\right] d\xi \leqslant \int_0^1 \left[\int_s^{s+h} G(x, t;\ \xi, \tau)\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} d\tau\right] d\xi .$$

From (7) and the Fubini's theorem we conclude that

$$(14)\quad \int_0^1 \left[\int_s^{s+h} G(x, t;\ \xi, \tau)\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} d\tau\right] d\xi = \int_s^{s+h} \left[\int_0^1 G(x, t;\ \xi, \tau)\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} d\xi\right] d\tau .$$

It follows from (6) that

$$(15)\qquad J \stackrel{\mathrm{df}}{=} \int_0^1 G(x, t;\ \xi, \tau)\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} d\xi$$

$$= G(x, t;\ \xi, \tau)\frac{\partial u(\xi, \tau)}{\partial \xi}\Big|_0^1 - \int_0^1 \frac{\partial G(x, t;\ \xi, \tau)}{\partial \xi}\,\frac{\partial u(\xi, \tau)}{\partial \xi} d\xi$$

for almost all τ. Using (2), integrating by parts, and applying (8) and (15), we get

$$(16)\qquad J = \int_0^1 \frac{\partial^2 G(x, t;\ \xi, \tau)}{\partial \xi^2} u(\xi, \tau) d\xi$$

for almost $\tau < t$.

It follows from (12)-(14) and from (16) that

$$\Delta_h \leqslant \int_s^{s+h} \left\{\int_0^1 \left(\frac{\partial G(x, t;\ \xi, \tau)}{\partial \tau} + \frac{\partial^2 G(x, t;\ \xi, \tau)}{\partial \xi^2}\right) u(\xi, \tau) d\xi\right\} d\tau .$$

By (3) we conclude therefore that $\Delta_h \leqslant 0$. In other words,

$$v(x, t;\ s+h) \leqslant v(x, t;\ s) \quad \text{if} \quad 0 < s < s+h < t .$$

By (1), (4) and (8) we have that $v(x, t;\ 0) = \int_0^1 G(x, t;\ \xi, 0) u(\xi, 0) d\xi \leqslant 0$, and consequently $v(x, t;\ h) \leqslant 0$. Using (4) we get therefore $u(x, h) = \lim_{t \to h+} v(x, t;\ h) \leqslant 0$.

Corollary. *It follows from our theorem that the unique function $u(x, t)$ which has the properties (5)—(7) and satisfies almost everywhere the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ and the homogeneous boundary conditions $u(1, t)$*

$= u(0,t) = 0$ *for* $0 \leqslant t \leqslant T$, $u(x,0) = 0$ *for* $0 \leqslant x \leqslant 1$ *is the function identically equal to zero.*

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MATHEMATICS

Analytic Solutions of the Eikonal Equation

by

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Presented by T. WAŻEWSKI on August 28, 1959

With a view to applying in asymptotic expansions of solutions of the Helmholtz equation, we will consider real functions of two variables $L(\boldsymbol{x})$, where $\boldsymbol{x}$ is a positional vector with polar co-ordinates r and ϑ, satisfying the equation

$$(\nabla L)^2 = 1 \tag{1}$$

in a domain D lying outside the curve C defined by the equation $r = f(\vartheta)$, where the function $f(\vartheta)$ is continuous and positive, with a period 2π. We assume also that the curve C is convex.

The principal aim of our paper is the demonstration that the eikonal constructed as the envelope of the family of planes has certain analytical properties.

The complete integral of (1) has the form

$$\boldsymbol{v}\cdot\boldsymbol{x} + b\,, \tag{2}$$

where $\boldsymbol{v}$ is a unit vector and b an arbitrary constant.

We form a one parameter family of solutions

$$\boldsymbol{v}(a)\cdot\boldsymbol{x} + b(a)\,, \tag{3}$$

where the vector $\boldsymbol{v}$ has polar co-ordinates 1 and a, and the function $b(a)$ is analytic, regular at real points, assumes real values at such points, and has a period 2π. The well known method of constructing solutions of the Eq. (1) consists in forming an envelope of a family of planes (3). In other words, the function L is obtained by eliminating the parameter a in (3) by means of the equation

$$\boldsymbol{v}'(a)\cdot\boldsymbol{x} + b'(a) = 0 \tag{4}$$

which can also be written in the form

$$r\sin(\vartheta - a) + b'(a) = 0\,. \tag{4'}$$

Now let us investigate the Eq. (4). For a given α this equation defines a straight line parallel to the vector $\boldsymbol{v}$ and lying at distance $|b'(\alpha)|$ from the origin. If we limit ourselves to ϑ satisfying the condition

$$|\vartheta-\alpha|<\frac{\pi}{2}. \tag{5}$$

we obtain a semiaxis designated in the sequel as $\Pi(\alpha)$ (Fig. 1).

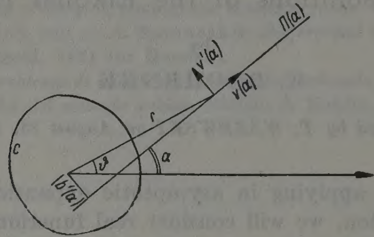

Fig. 1

The Eq. (4), with the condition (5) can also be considered from a different point. Namely, we can assume that this equation defines α as a function of $\boldsymbol{x}$, that is, to each point $\boldsymbol{x}$ values of α are assigned such that the semiaxis $\Pi(\alpha)$ passes through the point $\boldsymbol{x}$. We shall denote this function by $\alpha_0(\boldsymbol{x})$. In principle a function so defined may have several values.

THEOREM. A. *If the function $b(\alpha)$ satisfies the conditions specified above, and if, for each α from a closed interval $\langle 0, 2\pi\rangle$, the inequality*

$$-f\left(\alpha+\frac{\pi}{2}\right)<b'(\alpha)<f\left(\alpha-\frac{\pi}{2}\right), \tag{6}$$

holds, then the function $\alpha_0(\boldsymbol{x})$ is defined (not necessarily in a unique manner) for every point $\boldsymbol{x}$ from $D+C$. B. *If, moreover, for every α from the closed interval $\langle 0, 2\pi\rangle$ the following inequality is satisfied *) at the intersection point $\boldsymbol{x}$ of the semiaxis $\Pi(\alpha)$ with the curve C **)*

$$\lambda \stackrel{\text{df}}{=} \boldsymbol{v}(\alpha)\cdot\boldsymbol{x}-b''(\alpha)\geqslant\lambda_0>0\,, \quad \boldsymbol{x}\in\Pi(\alpha) \tag{7}$$

where λ_0 is constant, then the function $\alpha_0(\boldsymbol{x})$ is uniquely defined in the whole domain D and is in this domain an analytic function of the variables r and ϑ.

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*) Dr R. M. Lewis from New York has remarked that for better understanding of this paper it would be helpful to note that λ is the distance from the caustic whose equation is

$$\boldsymbol{\xi}=\boldsymbol{v}b''-\boldsymbol{v}'b'$$

**) Under our assumptions there exists only one such point.

Proof of A. Observe that, due to the inequality (6), the origin of the semiaxis $\Pi(a)$ for every a lies within the curve C. With a changing by 2π, our semiaxis makes a full revolution and must, therefore, pass at least once through every point $\boldsymbol{x}$ from $D+C$ *).

Proof of B. The condition (7) is similar to that of the classical theorem concerning implicit functions. The difference lies in the fact that we require our condition to be satisfied over the whole curve C, and obtain not a local but an integral theorem of uniqueness. Observe that condition (7), being satisfied at the point $\boldsymbol{x}_c$ holds also at all points of the semiaxis $\Pi(a)$ belonging to D. In fact λ, up to an additive constant, is the distance counted along the semiaxis from its origin; thus λ assumes greater values in D than on the curve C.

Consider a circle Γ with the centre at the origin of the system completely containing the curve C. It follows from conditions (4) and (5) that the point of intersection of the semiaxis $\Pi(a)$ with the circle defines the single-valued function $\vartheta(a)$. It follows from condition (7) and from conclusions drawn from it, that it is a monotonic function; hence an inverse function exists, too, and thus the function $a_0(\boldsymbol{x})$ is uniquely defined at points along the periphery of that circle. In order to prove that the function $a_0(\boldsymbol{x})$ is uniquely defined at the remaining points of D, assume that at a certain point belonging to D with co-ordinates r_0, ϑ_0 two semiaxes $\Pi(a_1)$ and $\Pi(a_2)$ intersect, where $a_2 > a_1$. Now consider the point of intersection of a semiaxis $\Pi(a_2)$ with the semiaxis $\Pi(a)$, where $a_1 \leqslant a < a_2$. The co-ordinates of this point, r and ϑ, are given by the system of equations

$$F_1 \overset{\mathrm{df}}{=} r\sin(\vartheta - a) + b'(a) = 0\,, \tag{8}$$

$$F_2 \overset{\mathrm{df}}{=} r\sin(\vartheta - a_2) + b'(a_2) = 0\,. \tag{9}$$

The Jacobian of this system equals

$$r\sin(a_2 - a) \tag{10}$$

and the derivative of the function $r(a)$ equals

$$r'(a) = \frac{\cos(\vartheta - a_2)}{\sin(a_2 - a)}[r\cos(\vartheta - a) - b''(a)]\,. \tag{11}$$

It is thus evident that, when a increases, beginning with a_1 the intersection point is pushed back along the semiaxis $\Pi(a_2)$. Due to the convexity of curve C this point will remain in D and with increasing a, will, as follows from the relation

$$r = r_0 + \int\limits_{a_1}^{a} r'(a')\,da' > r_0 + \lambda_0\cos(\vartheta_0 - a_2) \int\limits_{a_1}^{a} \frac{da'}{\sin(a_2 - a')}\,, \tag{12}$$

tend to infinity.

*) This can also be easily proved analytically.

For certain α, the semiaxes $\Pi(\alpha)$ and $\Pi(\alpha_2)$ will thus intersect on the periphery of a circle Γ which, as we know, is impossible. Thus the uniqueness of the function $\alpha_0(\boldsymbol{x})$, is proved.

To prove that the function $\alpha_0(\boldsymbol{x})$ is analytic, let us observe that, from the condition (7), it follows that at an arbitrary point $\boldsymbol{x}$ belonging to D

$$\boldsymbol{v}''[\alpha_0(\boldsymbol{x})]\cdot\boldsymbol{x}+b''[\alpha_0(\boldsymbol{x})] \leqslant -\lambda_0 < 0 \,. \tag{13}$$

Taking into consideration the analytic character of (4) it is obvious that, on the basis of the classical theorem concerning implicit functions the function $\alpha_0(\boldsymbol{x})$ is analytic.

Let us now substitute the function $\alpha_0(\boldsymbol{x})$ for α in (3). The function $L(\boldsymbol{x})$ so obtained will satisfy (1) in the domain D, and will be an analytic function of r and ϑ. The semiaxes $\Pi(\alpha)$ are in D projections of the characteristics and lines of the field ∇L.

The analytic functions $\alpha_0(\boldsymbol{x})$ and $L(\boldsymbol{x})$ defined in D in a unique manner, constitute a natural system of orthogonal curvilinear co-ordinates in the domain D. These functions define in a unique manner the position of the point in the domain in question. The lines $\alpha_0 = \text{const}$ are semiaxes. The lines $L = \text{const}$ constitute a family of curves orthogonal to the family previously discussed. In fact, it follows from (3) and (4) that

$$\nabla L = \boldsymbol{v}(\alpha_0)\,, \tag{14}$$

$$\nabla \alpha_0 = \frac{\boldsymbol{v}'(\alpha_0)}{\lambda}\,, \tag{15}$$

where, in accordance with (7)

$$\lambda = \boldsymbol{v}(\alpha_0)\cdot\boldsymbol{x} - b''(\alpha_0)\,. \tag{16}$$

It should be mentioned here that along a given semiaxis $\Pi(\alpha)$ both λ and L differ only by a constant from the distance to the origin of this semiaxis. A system of co-ordinates α_0 and λ could also be introduced; such a system would, however, not be in general an orthogonal one.

For the applications of the results arrived at in the present work, the functions $\nabla^2 L$ and $\nabla^2_{\alpha 0}$ should also be known. They are

$$\nabla^2 L = \frac{1}{\lambda} \tag{17}$$

and

$$\nabla^2 \alpha = \frac{b'(\alpha_0)+U'''(\alpha_0)}{\lambda^3} \tag{18}$$

respectively.

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MATHEMATICS

Some Remarks on Modular Spaces

by

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1. In [3] we have considered modulars and pseudomodulars $\varrho(x)$, defined as follows. Let X be a linear space. A functional $\varrho(x)$ defined in X and such that $-\infty < \varrho(x) \leqslant +\infty$ is called a *pseudomodular*, if

A.1′ $\quad \varrho(0) = 0$; $\quad$ A.2 $\quad \varrho(-x) = \varrho(x)$;

A.3 $\quad \varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y) \quad$ for $\quad x, y \in X,\ \alpha, \beta \geqslant 0,\ \alpha + \beta = 1$.

Evidently, $\varrho(x) \geqslant 0$. If

A.1 $\quad \varrho(x) = 0$ is equivalent to $x = 0$,

then the pseudomodular $\varrho(x)$ is called a *modular*. The following conditions are of importance:

B.1 $\quad$ if $\quad a_n \to 0$, $\quad$ then $\quad \varrho(a_n x) \to 0$;

B.2 $\quad$ if $\quad \varrho(x_n) \to 0$, $\quad$ then $\quad \varrho(2x_n) \to 0$.

In [3] we used the following notation:

$$X_\varrho = \{x \in X\colon\ \varrho(x) < +\infty\},\ X_\varrho^* = \{x \in X\colon\ \varrho(kx) < +\infty \text{ for some } k > 0\},$$
$$\bar{X}_\varrho^* = \{x \in X\colon\ x \text{ satisfies B.1}\}.$$

Then $\bar{X}_\varrho^* \subset X_\varrho^* \subset X$ and X_ϱ^* and $\bar{X}_\varrho^*$ are linear spaces. In X the following definitions of convergence and completeness are introduced:

(α) a sequence $\{x_n\} \subset X$ is *modular convergent* to $x \in X$, if $\varrho[k(x_n - x)] \to 0$ as $n \to \infty$ for some $k > 0$, dependent on $\{x_n\}$;

(β) a set $X_1 \subset X_\varrho^*$ is called *strongly modular complete*, if there exists a universal constant $k > 0$ such that for every sequence $\{x_n\} \subset X_1$ the condition $\varrho(x_n - x_m) \to 0$ as $m, n \to \infty$ implies $\varrho[k(x_n - x)] \to 0$ as $n \to \infty$, where $x \in X_1$.

1.1. In $\bar{X}_\varrho^*$ a topology may be introduced as follows. The sets $\{x \in \bar{X}_\varrho^*\colon\ \varrho(kx) < \varepsilon\}$, where $k > 0$ and $\varepsilon > 0$ form a base of neighbourhoods

of zero in $\bar{X}_\varrho^*$. Obviously, the topology induced by this base satisfies the first axiom of countability and hence it is equivalent to a norm topology. In [3] we defined this norm effectively by the formula

$$\text{(A)} \qquad \|x\| = \inf\{\varepsilon > 0:\ \varrho(x/\varepsilon) \leqslant \varepsilon\}\,.$$

This is an F-norm, when $\varrho(x)$ is a modular, and an F-pseudonorm, when $\varrho(x)$ is a pseudomodular. The convergence of a sequence of $x_n \in \bar{X}_\varrho^*$ to $x \in \bar{X}_\varrho^*$ in norm is equivalent to the convergence $\varrho[k(x_n - x)] \to 0$ for all $k > 0$ and equivalent to the convergence generated by the above defined topology. The normconvergence implies modular convergence to the same limit; the converse implication holds if and only if B.2 is satisfied in $\bar{X}_\varrho^*$. Moreover, if $\varrho(x)$ is a modular, then strong modular completeness implies completeness in norm of the space $\bar{X}_\varrho^*$ (cf. [3]).

2. Now, we show that in the special case of a convex $\varrho(x)$, the norm (pseudonorm) $\|x\|$ is equivalent to a B-norm (B-pseudonorm).

2.1. *If $\varrho(x)$ is a modular (pseudomodular), satisfying the convexity condition*

$$\varrho(\alpha x + \beta y) \leqslant \alpha\varrho(x) + \beta\varrho(y) \quad \textit{for} \quad x, y \in X\,,\ \alpha, \beta \geqslant 0\,,\ \alpha + \beta = 1\,,$$

then

$$\|x\|_* = \inf\{\varepsilon > 0:\ \varrho(x/\varepsilon) \leqslant 1\}$$

is a B-*norm* (B-*pseudonorm*) *and the following inequalities hold*:

$$\textit{if} \quad \|x\| \leqslant 1 \quad \textit{or} \quad \|x\|_* \leqslant 1\,, \quad \textit{then} \quad \|x\|_* \leqslant \|x\| \leqslant \sqrt{\|x\|_*}\,,$$

$$\textit{if} \quad \|x\| \geqslant 1 \quad \textit{or} \quad \|x\|_* \geqslant 1\,, \quad \textit{then} \quad \sqrt{\|x\|_*} \leqslant \|x\| \leqslant \|x\|_*\,.$$

Since the fact that $\|x\|_*$ is a B-norm (B-pseudonorm) is known, we shall prove only the inequalities. First, let us assume that $\|x\| \geqslant 1$ and take $\varepsilon > \|x\|$. Then $\varrho(x/\varepsilon) \leqslant 1$ implies $\varrho(x/\varepsilon) \leqslant \varepsilon$ and hence $\|x\| \leqslant \|x\|_*$. Conversely, since $\varrho(x/\varepsilon) \leqslant \varepsilon$, the convexity of $\varrho(x)$ yields $\varrho(x/\varepsilon^2) \leqslant \varrho(x/\varepsilon)/\varepsilon \leqslant 1$ and $\|x\|_* \leqslant \|x\|^2$. Now, let $\|x\| < 1$ and let $\|x\| < \varepsilon < 1$. Then $\varrho(x/\varepsilon) \leqslant \varepsilon$ implies $\varrho(x/\varepsilon) \leqslant 1$ and hence $\|x\|_* \leqslant \|x\|$. Taking $\eta > 0$ such that $\varrho(x/\eta) \leqslant 1$, we obtain $\varrho(x/\sqrt{\eta}) \leqslant \sqrt{\eta}\,\varrho(x/\eta) \leqslant \sqrt{\eta}$ and thus $\|x\| \leqslant \sqrt{\|x\|_*}$. It follows, that $\|x\|_* \leqslant 1$ implies $\|x\| \leqslant 1$ and $\|x\|_* \geqslant 1$ implies $\|x\| \geqslant 1$, and the proof is completed. It is easily seen that the inequalities between $\|x\|$ and $\|x\|_*$ are the best possible. Indeed, if $X = R^1$ = the space of reals, then for $\varrho(x) = |x|$, $\|x\| = \sqrt{\|x\|_*}$ and for $\varrho(x) = |x|^{a_n}$, where $a_n \to \infty$, $\|x\| = \|x\|_*^{a_n/(1+a_n)} \to \|x\|_*$ as $n \to \infty$.

3. Let $\varrho(x)$ be a pseudomodular (in general non-convex) and let us denote

$$X_0 = \{x \in X:\ \varrho(kx) = 0 \text{ for all } k\}\,.$$

Condition A.3 implies X_0 to be a linear subspace of X.

3.1. *There holds* $X_0 \cap X_\varrho^* = \{x \in X_\varrho^*: \|x\| = 0\}$.

Indeed, let $x \in X_\varrho^*$, $\|x\| = 0$; then $\varrho(x/\varepsilon) \leqslant \varepsilon$ for any $\varepsilon > 0$. If we fix an $\varepsilon > 0$ and take $|k| \geqslant 1/\varepsilon$, then $\varrho(kx) \leqslant \varepsilon$. But for $0 \leqslant |k| \leqslant 1/\varepsilon$, $\varrho(kx) \leqslant \varrho(x/\varepsilon) \leqslant \varepsilon$. Hence, $\varrho(kx) = 0$ for all k and $x \in X_0$. Since the converse inclusion is evident, we obtain the theorem.

3.2. Obviously, $0 \in X_0$. Pseudomodulars, for which X_0 contains only the element 0 will be called in the sequel *semimodulars*. Semimodulars are used in the theory of modular lattices (see e. g. [4], where convex semimodulars are considered). Let us note, that 3.1 implies immediately, that *if* $\varrho(x)$ *is a semimodular, then* $\|x\|$ *defined by* (A) *is a norm.*

3.3. Now, let us take a pseudomodular $\varrho(x)$, defined in X and let us take a linear subspace $X_1 \subset X$ such that the following condition

(B) $\varrho(\alpha x)$ is a left-side continuous function of α at $\alpha = 1$,

is satisfied for any $x \in X_1$, and $X_0 \subset X_1$. For instance, if (B) holds in $\bar{X}_\varrho^*$, one can take $X_1 = \bar{X}_\varrho^*$. Denote by $\widetilde{X}_1$ the quotient space $\widetilde{X}_1 = X_1/X_0$.

3.31. *If* $\widetilde{x} \in \widetilde{X}_1$ *and* $x_1, x_2 \in \widetilde{x}$, *then* $\varrho(x_1) = \varrho(x_2)$.

For arbitrary $0 < \alpha < 1$, we have $\alpha x_1 = \alpha x_2 + (1-\alpha)\alpha(x_1 - x_2)/(1-\alpha)$ and A.3 implies

$$\varrho(\alpha x_1) \leqslant \varrho(x_2) + \varrho\left[\frac{\alpha}{1-\alpha}(x_1 - x_2)\right].$$

But $x_1 - x_2 \in X_0$ and hence the second term at the right-hand side of the last inequality equals 0. Thus $\varrho(\alpha x_1) \leqslant \varrho(x_2)$ for any $0 < \alpha < 1$. Consequently, (B) implies $\varrho(x_1) \leqslant \varrho(x_2)$. Writing x_1 in place of x_2 and x_2 in place of x_1, we obtain $\varrho(x_2) \leqslant \varrho(x_1)$; hence, $\varrho(x_1) = \varrho(x_2)$.

3.32. *The formula* $\widetilde{\varrho}(\widetilde{x}) = \varrho(x)$, *where* x *is an arbitrary element of* $\widetilde{x}$, *defines a modular in* $\widetilde{X}_1$. *If* $\varrho(x)$ *is a semimodular, then* $\widetilde{X}_1 = X_1$.

This theorem follows from 3.31 and from the fact that $\widetilde{\varrho}(\widetilde{x}) = 0$ implies $x \in X_0$ for any $x \in \widetilde{x}$.

4. Now, we shall investigate some questions, concerning linear functionals over modular spaces. One can consider two classes of linear functionals: continuous with respect to the modular convergence (or shortly *ϱ-continuous*) and continuous with respect to the norm (A). Obviously, the first class is contained in the second one, but not conversely. As a counter example, let us take a space L_M^* with a convex $M(u)$, non-satisfying the condition (Δ_2) for large u. If $\bar{E}_M^*$ is the closure of the set of all bounded measurable functions in L_M^*, then it is known, that $L_M^* - \bar{E}_M^* \neq 0$ (see [2], Theorem 10.1). Let $x_0 \in L_M^* - \bar{E}_M^*$. Take a linear functional ξ, continuous with respect to the norm and such that $\xi(x) = 0$ for $x \in \bar{E}_M^*$ and $\xi(x_0) = 1$. This functional cannot be ϱ-continuous, since it equals zero in the set of bounded functions, and it is easily seen that bounded functions are dense in L_M^* with respect to the modular convergence.

4.1. The existence of linear functionals, continuous with respect to the norm convergence and discontinuous with respect to the modular convergence for every L_M^*-space with convex $M(u)$, non-satisfying (Δ_2) for large u, will follow also from Theorem 4.11 according to the known fact, that all linear functionals over L_M^*, continuous with respect to the norm have integral representation if and only if (Δ_2) holds for large u.

4.11. *Let μ be a finite, σ-additive measure, defined on a σ-algebra $\mathcal{F}$ of subsets of an abstract set E. Let $M(u)$ be an even, convex, continuous function defined for $-\infty < u < +\infty$, with $M(0) = 0$, $M(u) > 0$ for $u > 0$, $M(u)/u \to 0$ as $u \to 0$, $M(u)/u \to \infty$ as $u \to \infty$. Denote by $N(u)$ the function, complementary to $M(u)$ and put $\varrho(x) = \int_E M[x(t)]\,d\mu$ for all measurable functions $x(t)$; let $\bar{X}_\varrho^* = L_M^*$. The general form of ϱ-continuous linear functionals over L_M^* is $\xi(x) = \int_E x(t)y(t)\,d\mu$ with $y \in L_N^*$.*

First, we prove that $\xi(x) = \int_E x(t)y(t)\,d\mu$ is ϱ-continuous for any $y \in L_N^*$. If $y(t)$ is bounded, the ϱ-continuity of $\xi(x)$ follows easily from the condition $M(u)/u \to \infty$ as $u \to \infty$. In the general case we use the fact that bounded functions are dense in L_N^* with respect to modular convergence. Hence, if $y \in L_N^*$, then there exists for every $\varepsilon > 0$ a y_0 bounded such that $\int_E N\{k[y(t) - y_0(t)]\}\,d\mu < \varepsilon$. Writing $\xi_0(x) = \int_E x(t)y_0(t)\,d\mu$ we find that $\xi(x) - \xi_0(x) \leqslant (\varrho(x) + \varepsilon)k^{-1}$ for any $x \in L_M^*$, and the ϱ-continuity of $\xi(x)$ follows easily.

Now, take any ϱ-continuous linear functional $\xi(x)$ over L_M^*. Thus, $\xi(x)$ is continuous in norm and $|\xi(x)| \leqslant K\|x\|_*$. We shall now apply usual arguments, used for instance in [2], Theorem 14.2. We shall write $\xi(\chi_A) = \varphi(A)$, where χ_A is the characteristic function of the set A. Since the set function $\varphi(A)$ is σ-additive and absolutely continuous with respect to the measure μ, there exists by the Radon-Nikodym theorem a μ-integrable function $y(t)$ on E such that $\varphi(A) = \int_A y(t)\,d\mu$ for all $A \in \mathcal{F}$; hence, we have $\xi(x) = \int_E x(t)y(t)\,d\mu$ for all simple functions $x(t)$. Assuming $y(t) \geqslant 0$ we shall prove that $y \in L_N^*$. Take $y_n(t) = y(t)$ for $y(t) \leqslant n$ and $y_n(t) = n$ for $y(t) > n$ and take a bounded function $x_n(t) \geqslant 0$ such that $\int_E x_n(t)y_n(t)\,d\mu/K = \varrho(x_n) + \int_E N[y_n(t)/K]\,d\mu$. We may assume that $\varrho(x_n) \leqslant 1$ for almost all n; in another case we should have $\int_E x_n(t)y_n(t)\,d\mu/K \leqslant \varrho(x_n)$ and $y(t) = 0$ almost everywhere. Hence $\|x_n\|_* \leqslant 1$ for almost all n and $\int_E x_n(t)y_n(t)\,d\mu/K \leqslant 1$, but this yields $\int_E N[y_n(t)/K]\,d\mu \leqslant 1$, and by Fatou's lemma, $\int_E N[y(t)/K]\,d\mu \leqslant 1$ i. e. $y \in L_N^*$. Simple functions being dense in L_M^* with respect to modular convergence, this gives $\xi(x) = \int_E x(t)y(t)\,d\mu$ for all $x \in L_M^*$. The general case may be always reduced to the case $y(t) \geqslant 0$.

4.2. The above theorems show that investigations of ϱ-continuous linear functionals are of interest. We shall prove some theorems about the existence of such non-trivial functionals in the case of modular lattices. Results of this type for concave modulars were obtained also in [5], and for L_M^* with $M(u)$ satisfying (Δ_2) for large u, in [6].

5. Let us take a *linear lattice* X (for definition, see e. g. [4], where such a space is called a semi-ordered linear space) with lattice operations $\vee$ (supremum) and $\wedge$ (infimum), and a functional $\varrho(x)$, defined in X, such that $-\infty < \varrho(x) \leqslant +\infty$ and satisfying the following conditions:

(1) $\varrho(kx) = 0$ for all $k > 0$, if and only if $x = 0$;

(2) $\varrho(kx) < +\infty$ for some $k > 0$, dependent on x;

(3) $|x| \leqslant |y|$ implies $\varrho(x) \leqslant \varrho(y)$;

(4) $x \wedge y = 0$ implies $\varrho(x+y) = \varrho(x) + \varrho(y)$;

(5) $\varrho(x) = \varrho(|x|)$;

(6) $x_n \geqslant 0$ implies the existence of $\inf\limits_{n} x_n$.

Such functionals $\varrho(x)$ with additional assumption of convexity of $\varrho(kx)$ in k for every x, are considered in [4]. We shall use also the following conditions:

(7) $$0 \leqslant x_\lambda \uparrow x_0 \quad \text{implies} \quad \varrho(x_0) = \sup_\lambda \varrho(x_\lambda);$$

(8) if $x_1 \leqslant \ldots \leqslant x_\lambda \leqslant \ldots \leqslant x$ and $\varrho(x) < +\infty$, then there exists $\sup\limits_\lambda x_\lambda$.

Here $x_\lambda \uparrow x_0$ denotes that x_λ form an ordered set with supremum x_0.

5.1. *If* (7) *and* (8) *are satisfied only for enumerable sequences of x_λ and if $0 \leqslant x < y$ implies $\varrho(x) < \varrho(y)$, then* (7) *and* (8) *hold for an arbitrary sequence x_λ too.*

Denote $s = \sup\limits_\lambda \varrho(x_\lambda)$ and assume $\varrho(x_\lambda) < s$ for all λ. Take a numerical sequence s_n, increasing to s, $s_0 = 0$. Further, denote $\Lambda_n = \{\lambda: s_{n-1} \leqslant \varrho(x_\lambda) < s_n\}$ and take $\lambda_n \in \Lambda_n$. Obviously, $\sup\limits_n \varrho(x_{\lambda_n}) = s$ and writing $x_0 = \sup\limits_n x_{\lambda_n}$, we have $x_\lambda \leqslant x_0$ for all λ; it is obvious, that if $y \geqslant x_\lambda$ for all λ, then $y \geqslant x_{\lambda_n}$ for all n and, consequently, $y \geqslant x_0$. Hence, $x_0 = \sup\limits_\lambda x_\lambda$ and $\varrho(x_0) = \sup\limits_\lambda \varrho(x_\lambda)$. If $\varrho(x_\lambda) = s$ for some λ_0, then $x_{\lambda_0} = \sup\limits_\lambda x_\lambda$ and (7) and (8) are obvious.

5.2. *If $\varrho(x)$ satisfies* (1)-(6), *then $\varrho(x) \geqslant 0$ and*

$$\varrho(x \vee y) + \varrho(x \wedge y) = \varrho(x) + \varrho(y) \quad \textit{for} \quad x, y \geqslant 0\,.$$

The inequality $\varrho(x) \geqslant 0$ follows from (5), (3) and (1) and the equality is known.

5.3. *If $\varrho(x)$ satisfies* (1)-(6), *then $\varrho(x)$ is a semimodular.*

We have only to prove A.3. Take $x, y \geqslant 0$. Since $x \leqslant x \vee y$ and $y \leqslant x \vee y$, we have $\alpha x + \beta y \leqslant x \vee y$ for any $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$. Hence, (3) implies $\varrho(\alpha x + \beta y) \leqslant \varrho(x \vee y) \leqslant \varrho(x) + \varrho(y)$. Applying (5) and (3), we obtain A.3 for arbitrary $x, y \in X$.

5.31. It follows from 5.3, that if X is a linear lattice with a functional $\varrho(x)$, satisfying (1)-(6) and such that $\varrho(x) = 0$ implies $x = 0$, then $X_\varrho^* = X$, and the F-norm $\|x\|$ defined in (A) may be introduced in $\bar{X}_\varrho^*$. Moreover, $\bar{X}_\varrho^*$ is also a linear lattice, i. e. if $x, y \in \bar{X}_\varrho^*$, then $x \vee y \in \bar{X}_\varrho^*$ and $x \wedge y \in \bar{X}_\varrho^*$. This follows from the fact that $x \vee y$ and $x \wedge y$ are linear combinations of x, y and $|x-y|$. A linear lattice $\bar{X}_\varrho^*$ with a modular $\varrho(x)$, satisfying (1)-(8), will be called a *modular lattice*. If $\varrho(x)$ is a semimodular only, the modular lattice $\bar{X}_\varrho^*/X_0$ may be considered.

5.4. A modular lattice $\bar{X}_\varrho^*$ will be called *non-atomic*, if for every $x \in \bar{X}_\varrho^*$, $x > 0$, there exist $y, z \in \bar{X}_\varrho^*$ such that $x = y + z$, $y \wedge z = 0$, $y, z > 0$.

5.41. *If a modular lattice $\bar{X}_\varrho^*$ is non-atomic, then for every $x \in \bar{X}_\varrho^*$, $x > 0$ and for each positive integer n there exist 2^n positive elements $y_1, y_2, \ldots, y_{2^n} \in \bar{X}_\varrho^*$ such that $x = y_1 + y_2 + \ldots + y_{2^n}$, $y_i \wedge y_j = 0$ for $i \neq j$ and $\varrho(y_i) = 2^{-n}\varrho(x)$ for $i = 1, 2, \ldots, 2^n$.*

This known lemma may be proved by application of Zorn's lemma to the set of all $y > 0$ such that $y \wedge (x-y) = 0$, $x - y > 0$, $0 < \varrho(y) \leqslant \varrho(x)/2$.

6. The following theorem on non-existence of non-trivial ϱ-continuous linear functionals holds.

6.1. *Let $\bar{X}_\varrho^*$ be a non-atomic modular lattice and let*

$$\lim_{k \to \infty} \varrho(kx)/k = 0 \tag{C}$$

in a set of $x \geqslant 0$, modular-dense in the positive cone of $\bar{X}_\varrho^*$. The set of ϱ-continuous linear functionals over $\bar{X}_\varrho^*$ consists only of the trivial functional.

To prove this theorem, take a convex and symmetric set W such that $\varrho(y) \leqslant \varepsilon$ implies $y \in W$ for some $\varepsilon > 0$. Take an $x \in \bar{X}_\varrho^*$, $x > 0$ such that (C) holds. We shall prove that $x \in W$. There exists an increasing sequence of positive integers k_n such that $2^{-k_n}\varrho(2^{k_n}x) \leqslant \varepsilon$. Given n, we apply 5.41 to $2^{k_n}x$. We find $2^{k_n}x = y_1 + y_2 + \ldots + y_{2^{k_n}}$, where $y_1, y_2, \ldots, y_{2^{k_n}} \in \bar{X}_\varrho$, $y_i \wedge y_j = 0$ for $i \neq j$, $\varrho(y_i) = 2^{-k_n}\varrho(2^{k_n}x)$. Hence, $y_i \in W$ for $i = 1, 2, \ldots, 2^{k_n}$; thus $x = 2^{-k_n}(y_1 + y_2 + \ldots + y_{2^{k_n}}) \in W$. Now, assume that $\xi(x)$ is a ϱ-continuous linear functional over $\bar{X}_\varrho^*$ and define $W = \{x \in \bar{X}_\varrho^* : |\xi(x)| \leqslant 1\}$. Obviously, the assumptions on W are satisfied and W is closed with respect to the modular convergence. Hence W contains the positive cone of $\bar{X}_\varrho^*$; but this implies $\xi(x) = 0$ for all $x \geqslant 0$, $x \in \bar{X}_\varrho^*$ and thus for all $x \in \bar{X}_\varrho^*$.

6.2. As examples of the above theorem we shall prove some theorems on spaces L_M^*. Let μ satisfy the same assumptions as in 4.11. Assume

that μ is *non-atomic*, i. e. for every $A \in \mathcal{F}$ with $\mu A > 0$ there exist $B, C \in \mathcal{F}$ such that $A = B \cup C$, $B \cap C = 0$, $\mu B > 0$, $\mu C > 0$. We take a function $M(u, v)$, defined in $E \times R^1$ (R^1 = the space of reals), $M(u, v) \geqslant 0$ and $M(u, v) = 0$ if and only if $v = 0$. Let $M(u, v)$ be an even, continuous and increasing (for $v \geqslant 0$) function of v, for every $u \in E$, and an integrable function in E for every $v \in R^1$. It is known that $\varrho(x) = \int_E M[t, x(t)]\, d\mu$ is a modular (see [3], pp. 60-63).

6.21. *By the usual definition of order, $\bar{X}_\varrho^*$ with the above defined $\varrho(x)$ is a non-atomic modular lattice.*

We shall prove only (7) and (8). Take $0 \leqslant x < y$. Then $\varrho(x) < \varrho(y)$, since assuming $\varrho(x) = \varrho(y)$ we should have $x(t) = y(t)$ almost everywhere. Moreover, (7) and (8) are evident for enumerable sequences x_λ; hence, 5.1 implies (7) and (8) in the general case.

6.22. *If the measure μ and the function $M(u, v)$ are defined as in 6.2 and if $\varliminf_{v\to\infty} v^{-1} \int_E M(u, v)\, d\mu = 0$, then there exist over the space $\bar{X}_\varrho^*$ only trivial ϱ-continuous linear functionals.*

We have for $|x(t)| \leqslant s$,

$$\frac{1}{k_n} \varrho(k_n x) = \frac{1}{k_n} \int_E M[t, k_n x(t)]\, d\mu \leqslant$$

$$\leqslant \frac{1}{k_n} \int_E M(t, k_n s)\, d\mu = s \frac{1}{k_n s} \int_E M(t, k_n s)\, d\mu \to 0$$

for a certain sequence $k_n \to \infty$ and hence $\varliminf_{k\to\infty} \varrho(kx)/k = 0$ for all bounded functions $x(t)$; but bounded functions are dense in $\bar{X}_\varrho^*$ with respect to modular convergence and one can apply 6.1.

6.3. Let μ be a finite, non-atomic, σ-additive measure, defined on a σ-algebra of subsets of an abstract set E. Let $M(u)$ be an even, continuous function, defined for $-\infty < u < +\infty$ and increasing for $u \geqslant 0$, with $M(0) = 0$, $M(u) > 0$ for $u > 0$. Define in the set of all functions $x(t)$, measurable in E, a modular $\varrho(x) = \int_E M[x(t)]\, d\mu$ and denote $\bar{X}_\varrho^* = L_M^*$.

6.31. *Under the assumptions of 6.3, there exist over L_M^* non-trivial ϱ-continuous linear functionals if and only if $\varliminf_{u\to\infty} M(u)/u > 0$.*

Assuming $\varliminf_{u\to\infty} M(u)/u = 0$, according to 6.22 there exists only trivial ϱ-continuous functional over L_M^*. Now, assume $M(u) > cu$ for $u > u_0$, where $u_0, c > 0$ and take $\xi(x) = \int_E x(t)\, d\mu$; the existence of this integral for $x \in L_M^*$ is evident. Take a sequence of $x_n \in L_M^*$ such that $\int_E M[kx_n(t)]\, d\mu \to 0$, where $k > 0$. Then $x_n(t)$ tend to 0 in measure and for $|x_n(t)| > u_0$, $|x_n(t)| < M[x_n(t)]/c$. It follows that $\xi(x_n) \to 0$.

6.32. The above theorem yields a generalization of a theorem by Rolewicz [6]; if we assume that $M(u)$ satisfies the condition (Δ_2) for large u, norm convergence is equivalent to modular convergence in L_M^* (see [3])] and 6.31 gives the theorem 4 in [6].

7. To mention one of other theorems, which are true for modular lattices without the assumption of convexity of $\varrho(x)$, we shall write here only a theorem by Amemiya ([1], [7]), being in connection with the condition (Δ_2): *if X is a universally continuous, non-atomic linear lattice such that $X_\varrho = X$ and if $\varrho(x)$ is monotone complete, then for every $\varepsilon > 0$ there exists $\varkappa > 0$ such that $\varrho(2x) \leqslant \varkappa\varrho(x)$ for $\varrho(x) \geqslant \varepsilon$, i. e. $\varrho(x)$ is semi-upper bounded* (for terminology, see [4] or [7]).

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MATHÉMATIQUE

Sur la convolution par $\exp(t^2)$

par

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Présenté par A. MOSTOWSKI, le 14 septembre 1959

1. Soit L la classe des fonctions $f(t)$ sommables sur tout intervalle fini $0 \leqslant t \leqslant T$ pour lesquelles la transformée de Laplace

$$\int_0^\infty e^{-st} f(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt$$

est convergente pour certaines valeurs de s. Les fonctions de la classe L soient dites *transformables*.

Évidemment, la fonction e^{t^2} n'est pas transformable. Pour cette raison elle échappe au calcul opérationnel, basé sur la transformation de Laplace. Elle se laisse cependant traiter dans la théorie algébrique [1], où les opérateurs sont définis comme les quotients $\frac{p}{q}$ des fonctions continues, la division étant entendue comme l'opération inverse à la convolution. La question s'impose: la fonction e^{t^2} peut-elle être representée comme le quotient $\frac{p}{q}$ (au sens précédent) des deux fonctions p et q transformables?

La réponse est négative et peut être exprimée sous la forme du théorème suivant:

THÉORÈME. *Si* $f(t) \in L$ *et* $\int_0^t f(t-\tau)e^{\tau^2}\,d\tau \in L$, *on a* $f(t) = 0$ *presque partout dans* $0 \leqslant t < \infty$.

2. Démonstration. Soit

$$g(t) = \int_0^t f(t-\tau)e^{\tau^2}\,d\tau$$

et soit s un nombre positif pour lequel les transformées de Laplace de f et g sont convergentes. En posant

$$F(t) = \int_0^T e^{-st} f(t)\,dt \quad \text{et} \quad G(T) = \int_0^T e^{-st} g(t)\,dt\,,$$

nous obtenons qu'il existe un nombre M tel que

$$|F(T)| < M \quad \text{et} \quad |G(T)| < M\,.$$

Si $0 < U < T$, il est facile de vérifier que

$$G = G_1 + G_2\,,$$

où

$$G_1 = \int_0^{T-U} e^{-s\sigma} e^{\sigma^2} F(t-\sigma)\,d\sigma\,,$$

$$G_2 = \int_{T-U}^{T} e^{-s\sigma} e^{\sigma^2} F(T-\sigma)\,d\sigma\,.$$

On a

$$|G_1| < M e^{(T-U)^2} \int_0^{T-U} e^{-s\sigma}\,d\sigma < \frac{M}{s} e^{(T-U)^2}$$

et

$$G_2 = e^{T^2-sT} \cdot J\,,$$

où

$$J = \int_0^U e^{(s+2T)u} e^{u^2} F(u)\,du\,.$$

Il s'ensuit que

$$|J| = |G_2| e^{-T^2+sT} \leqslant (|G| + |G_1|) e^{-T^2+sT} \leqslant M e^{-T^2+sT} + \frac{M}{s} e^{U^2+(s-2U)T}\,.$$

Soit $U > \frac{1}{2}s$. Alors l'intégrale J est bornée pour $0 < T < \infty$, ce qui entraîne $e^{t^2} F(t) = 0$ $(0 \leqslant t \leqslant U)$, d'après un théorème connu des moments bornés (voir p. ex. [2], p. 18). Comme l'intervalle $0 \leqslant t \leqslant U$ peut être chosi arbitrairement grand, il s'ensuit que

$$\int_0^\infty e^{-st} f(t)\,dt = 0\,. \tag{1}$$

Cette égalité vient d'être démontrée pour tout s pour lequel les transformées de f et g convergent. Or, d'après l'hypothèse, elles convergent pour un certain $s = s_0$ et, par consequent, pour tout $s \geqslant s_0$. Donc, l'égalité (1) a lieu pour tout $s \geqslant s_0$, ce qui entraîne $f = 0$ presque partout dans $0 \leqslant t < \infty$.

3. Remarquons que l'hypothèse $f(t) \in L$ dans le théorème précédent ne peut pas être supprimée, c'est-à-dire que la condition $\int\limits_0^t f(t-\tau)e^{\tau^2}\,d\tau \in L$ seule n'entraîne pas $f(t) = 0$ presque partout. En effet, considérons l'équation intégrale

$$\int\limits_0^t f(t-\tau)e^{\tau^2}d\tau = t\,.$$

En posant

$$\{t\} = l^2 \quad \text{et} \quad \{e^{t^2}\} = l(1+a)\,, \quad \text{où} \quad a = \{2te^{t^2}\}\,,$$

cette équation s'écrit opérationnellement

$$f\cdot l(1+a) = l^2\,,$$

d'où

$$f = l(1-a+a^2-\ldots)\,.$$

Célà prouve que f est une fonction continue dans $0 \leqslant t < \infty$ (v. [2], p. 169).

D'après le théorème précédent, la fonction f n'est pas transformable.

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MATHÉMATIQUE

La construction du champ de probabilité avec la solution fondamentale de l'équation parabolique normale aux coefficients hölderiens

par

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Présenté par K. KURATOWSKI, le 26 septembre 1959

Dans les travaux antérieurs [2]-[4] la construction du champ continu de probabilité dans les processus stochastiques fût traitée en admettant les conditions de Hölder pour les coefficients de la 2-de équation de Kolmogoroff: $\overline{\psi}(u)=0$. On a démontré [1], avec les mêmes hypothèses sur les coefficients de la 1-ère équation de Kolmogoroff ((1,1), [1]): $\hat{\psi}(u)=0$, (à savoir les hypothèses I et II [1]), que sa solution fondamentale normée, $U(X,t;\ Y,\tau)$ ((1,2), [1]) est la probabilité de passage. A présent on se demande, si ces conditions suffisent pour la construction du champ. La reponse est affirmative.

On part de la definition du champ de probabilité de l'article [5] et de la définition de la fonction $U(X,t;\ Y,\tau)$. On prouve, *que la fonction de champ*

$$(1)\qquad f(X,t;\ Y,\tau)=f(X,t)\,U(X,t;\ Y,\tau);\ X\in R_n,\ Y\in R_n,\qquad 0\leqslant t<\tau\leqslant T=\text{const}$$

est la densité de la probabilité dans R_{2n} (ou *fonction de champ dans* R_{2n}) dès que la fonction $f(X,t)$ est la densité de la répartition dans R_n *) (ou *fonction de champ dans* R_n). On en déduit, que la fonction:

$$(2)\qquad \varphi(Y,t,\tau)=\int\limits_{R_n} f(u,t)\,U(u,t;\ Y,\tau)\,du\,,\quad t<\tau,\ Y\in R_n$$

est aussi une fonction de champ dans R_n **).

---

*) Cette répartition est une répartition marginale horizontale [5].
**) C'est la densité de la répartition marginale verticale [5].

Les fonctions $U(X,t;\ Y,\tau)$, $f(X,t)$ et $\varphi(Y,t,\tau)$ définissent complétement le champ dans R_{2n} par les formules:

$$(3)\qquad U^*(Y,\tau;\ X,t) = [f(X,t)/\varphi(Y,t,\tau)]\,U(X,t;\ Y,\tau)\,,$$

$$(4)\qquad \int\limits_{-\infty}^{Y} U(X,t;v,\tau)\,dv \quad \text{et} \quad \int\limits_{-\infty}^{X} U^*(Y,\tau;\ u,t)\,du\,.$$

Les intégrales (4) représentent les deux composantes du champ: la verticale et la horizontale respectivement. Leurs densités seront nommées *fonctions de passages verticale et horizontale*, si elles satisfont à l'équation de Smoluchowski. D'après [1], on n'a qu'à établir cette équation pour la composante horizontale. Observons dans ce but, qu'en général la fonction de champ marginale verticale $\varphi(Y,t,\tau)$ dépendra du temps $t<\tau$. Elle pourra être différente de la fonction $f(Y,\tau)$. Il n'est pas ainsi dès qu' on a supposé, comme dans les articles cités ([2]-[4]), que l'équation $\overline{\psi}(u)=0$ existe, et que la solution fondamentale $\Gamma(X,t;\ Y,\tau)$ de l'équation $\hat{\psi}(u)=0$ est en même temps la solution de l'équation $\overline{\psi}(u)=0$, à savoir, que:

$$(5)\qquad \overline{\psi}_{Y,\tau}\big(\Gamma(X,t;\ Y,\tau)\big)=0\,.$$

On peut démontrer, qu'on a alors:

$$(6)\qquad \varphi(Y,t,\tau)=f(Y,\tau)\,,$$

dès que $\overline{\psi}(f(Y,\tau)/A(Y,\tau))=0$, où $A(Y,\tau)=U(X,t;Y,\tau)/\Gamma(X,t;Y,\tau)\cdot\lambda$ ((1,2), [1]). Dans ce cas, les formules (1) et (3) séraient remplacées par:

$$(7)\qquad f(X,t;\ Y,\tau)=f(X,t)\,U(X,t;\ Y,\tau)=f(Y,\tau)\,U^*(Y,\tau;\ X,t)\,.$$

On peut démontrer, qu'il résulte de l'égalité (7) et de l'article [1], *que la fonction $U^*(Y,\tau;\ X,t)$ est une fonction de passage.* Le champ de probabilité dans R_{2n} défini par la formule (7) sera nommé *champ simple.*

Le bût de cette note est de prouver que les hypothèses I et II de [1] permettent de construire un champ simple. En effet, il résulte de la formule (2), que la formule (7) tiendra, si seulement la fonction $f(Y,\tau)$ est une solution de l'équation:

$$(8)\qquad f(Y,\tau)=\int\limits_{R_n} f(u,t)\,U(u,t;\ Y,\tau)\,du\,.$$

C'est l'équation de Fredholm de seconde espèce. Son domaine d'integration étant non borné, ne permet pas d'affirmer qu'elle possède une solution non nulle. Dans ce but nous faisons intervenir la seconde intégrale generalisée de Poisson-Weierstrass [6], [7] *), à savoir:

$$(9)\qquad I_1(Y,t,\tau)=\int\limits_{R_n} \varrho(u,t)\,\Gamma(u,t;Y,\tau)\,du\,,\quad t<\tau\,,$$

*) Il est évident, qu'au cas de l'hypothèse (5) on a: $\overline{\psi}_{Y,\tau}(I_1\subset Y,t,\tau)\equiv 0$.

où $\varrho(u, t)$ est une fonction intégrable et bornée dans R_n. On remplace l'intégrale $I_1(Y, t, \tau)$ par $\bar{I}(Y, t, \tau)$ en écrivant:

$$(10) \quad \bar{I}(Y, t, \tau) = \lambda I_1(Y, t, \tau) A(Y, \tau) = \int_{R_n} \varrho(u, t)\, U(u, t;\; Y, \tau)\, du\,.$$

On démontre, comme dans [6], [7], que si la fonction $\varrho(u, t)$ est continue au point (Y, t), alors $\lim_{\tau \to t} \bar{I}(Y, t, \tau) = \varrho(Y, t)$. Nommons par $f(u, 0)$ une fonction continue du champ dans R_n et la fonction $\bar{I}(Y, 0, \tau)$ correspondente par $f(Y, \tau)$, de sorte qu'on a:

$$(11) \quad f(Y, \tau) = \int_{R_n} f(u, 0)\, U(u, 0; Y, \tau)\, du\,.$$

Mais nous savons déjà, que:

$$(12) \quad f(Y, \tau) \to f(Y, 0),\ \tau \to 0,\ f(Y, \tau) > 0\,, \quad \int_{R_n} f(Y, \tau)\, dY = 1\,.$$

La fonction $f(Y, \tau)$ est donc une fonction de champ dans R_n. On y démontre que c'est la solution de l'équation integrale (8). D'ici résulte le

THÉORÈME. *Admettons les hypothèses I et II de l'ouvrage* [1]; *alors la solution fondamentale normée $U(X, t; Y, \tau)$ de la prémière équation de Kolmogoroff, $\hat{\varphi}(u) = 0$, la seconde intégrale généralisée de Poisson-Weierstrass de cette solution, pour $t = 0$, d'une fonction de champ, continue dans R_n', déterminent dans R_{2n} le champ simple.*

Ses deux composantes satisfont à l'équation de Smoluchowski.

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ASTRONOMIE

Methode der Untersuchung der Vertexrichtung von Sterngruppen mit geringer Mitgliedernanzahl auf Grund der Eigenbewegungen der Sterne

von

K. RUDNICKI

Vorgelegt von W. DZIEWULSKI am 15 September 1959

Methode

Wenn wir nur über Daten von Eigenbewegungen einer kleinen Sterngruppe verfügen, so ist es angewiesen, auf die Bestimmung der Parameter des Geschwindigkeitkörpers zu verzichten und sich auf die Untersuchung der Vertexrichtung zu beschränken. Man kann sich dabei einer Methode bedienen, die Bessel'schen Grundgedanken der Apexbestimmung folgt. Die obigen Grundgedanken stammen vom Jahre 1818 (vgl. [2]).

In den nachstehenden Bemerkungen schlagen wir eine Methode vor, die uns dem erwähnten Zweck adequat scheint.

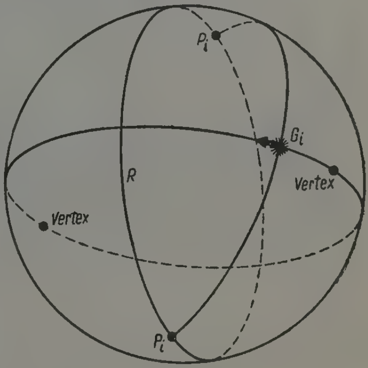

Abb. 1

Es seien jedem Sterne G_i (siehe Abbildung) die Punkte P_i auf der Himmelsphäre zugeordnet, die die Pole eines grossen Berührüngskreises zur Pekuliarkomponente der Eigenbewegung bilden. Sind die Pekuliargeschwindigkeiten der Sterne streng in derselben Direktion wie das Vertex gerichtet, so müssen die Punkte P_i auf einem grossen Kreis R Platz finden, dessen beide Pole die Vertizes bilden. In Wirklichkeit aber, ist das gesuchte Vertex ein Pol eines grossen Kreises, dessen entlang sich ein Streifen der grössten Verdichtung der P_i-Punkte befindet. Falls sich die Verdichtung der Punkte deutlich macht, kann die Vertexrichtung mit Hilfe

einer der Ausgleichungsmethoden bestimmt werden. Im Falle dagegen, in dem sich die P_i-Punkte ziemlich gleiehmässig zerstreuen, können wir die "Beweglichkeit" der Sterne in einzelnen Richtungen bestimmen, indem wir die Quadraten der Winkelentfernungen der P_i-Punkte von beliebig auf der Himmelsphäre gewählten grossen Kreisen untersuchen.

Beispiel der Anwendung

Ikaunieks hat eine Hypothese aufgeworfen [1], dass das Vertex der N-und R-Sterne – im grossen genommen – senkrecht zur galaktischen Ebene gerichtet ist. Um diese Hypothese zu überprüfen, nehmen wir als Ausgangspunkt unserer Betrachtungen die Daten über die Eigenbewegungen der 45 N- und R-Sterne, die im Bosschen Katalog GC angegeben sind. Alle Berechnungen wurden sowohl im dem GC als in dem $FK3$-System durchgeführt. Es wurden dabei die Standartdaten für galaktische Rotation (Oortsche Formeln für $\Delta\mu_l$ und $\Delta\mu_b$), sowie Standartdaten für die Apexbewegung in Rücksicht genommen; die letzten allerdings – da die Paralaxen immer unsicher sind – unter der Annahme, dass die Sternenentfernungen entsprechend 1/2-, 1- und 2-fache (Hypothesen I, II und III) der von Ikaunieks angenommen Grössen [1] ausmachen. Für die Hypothesen I und III wurde ausserdem nachträglich eine Rechnung in dem GC-System durchgeführt, die die galaktische Rotation ausser Acht lässt, die Schnellaüferapexdaten $D = +60°$, $A = 300°$, $V_\odot = 280$ km/sec dagegen berücksichtigt. Die Ausgleichrechnung konnte dabei nur für den Fall des Schnellaüferapexes angewendet werden, und die Resultate (vgl. Tafel) können völlig mit der Scheinbeweglichkeit der Sterne in der angenommenen Apexrichtung erklärt werden. Diese Scheinbeweglichkeit ist von dem allzugrossen angenommenen Wert $V_\odot$ hervorgerufen. Was die anderen Fälle anbetrifft, so konnte kein Ausgleichungprozess infolge der zu gleichmässigen Verstreuung der P-Punkte zu einem reelen Resultat führen.

Für alle Fälle wurden die K-Werte berechnet und zwar nach der Formel

$$K = \frac{\sum\limits_{i} \cos^2 d_{1i}}{\sum\limits_{i} \cos^2 d_{2i}},$$

wo d_{1i} die Winkelentfernung des P_i-Punktes vom galaktichen Pol bedeutet, d_{2i} – die Winkelentfernung des P_i-Punktes von dem Punkte $b^I = 0^0$ und $l^I = 0^0$ (im Lund Tables System). Ausserdem wurden auf Grund ähnlicher Formeln für beide Schnellaüferapexhypothesen die K'-Werte berechnet, wobei d_{2i} als eine Winkelentfernung des P_i-Punktes von der Vertexrichtung angenommen sein soll, die wir mittels der Ausgleichrechnung gewonnen haben.

Die Resultate unserer Errechnungen sind in der folgenden Tafel angegeben:

		Hypothese I		Hypothese II		Hypothese III	
		K	K'	K	K'	K	K'
Standartapex	GC System	1.02	—	1.09	—	1.48	—
	FK3 System	1.30	—	1.09	—	1.21	—
Schnelläuferapex		1.73	2.05	—	—	1.53	3.23
GC System	Ausgleichungsresultat	$l^{\mathrm{I}}=25°\pm 9°$ $b^{\mathrm{I}}=+1°\pm 6°$		—		$l^{\mathrm{I}}=58°\pm 9°$ $b^{\mathrm{I}}=+16°\pm 5°$	

Alle K- und K'-Werte sind grösser als 1, was folgendermassen zu interpretieren ist:

die Eigenbewegungen der N- und R-Sterne die in dem Katalog *GC* angegeben sind, geben keinen Anlass zu der Annahme, dass ihre Vertexrichtung senkrecht zu der galaktischen Ebene ist.

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ASTROPHYSICS

The Magnetogravitational Instability of the Medium of Finite Electrical Conductivity

by

A. G. PACHOLCZYK and J. S. STODÓŁKIEWICZ

Presented by W. RUBINOWICZ on September 18, 1959

Introduction

In the present paper we discuss the problem of the influence of finite isotropic electrical conductivity on the magnetogravitational instability of rotating medium. We make the same assumptions as in [1] *) with two exceptions: first, we assume that the medium is nonviscous ($\eta = 0$), and further we assume that the medium considered is characterized by the finite isotropic electrical conductivity ($\nu_m < +\infty$).

Basic equations

In the case of angular velocity of rotation perpendicular to the yz plane, the basic equations of the considered problem are:

$$(1) \quad \varrho\dot{\vec{u}} = -\operatorname{grad}\delta p - (4\pi)^{-1}\vec{h}\times\operatorname{rot}\vec{H} - (4\pi)^{-1}\vec{H}\times\operatorname{rot}\vec{h} + 2\varrho\vec{u}\times\vec{\Omega} + \varrho\operatorname{grad}\delta\psi,$$

$$(2) \quad \dot{\vec{h}} = \operatorname{rot}(\vec{u}\times\vec{H}) + \nu_m\Delta\vec{h},$$

$$(3) \quad \operatorname{div}\vec{h} = 0,$$

$$(4) \quad \delta\dot{\varrho} + \operatorname{div}\varrho\vec{u} = 0,$$

$$(5) \quad \Delta\,\delta\psi + 4\pi G\,\delta\varrho = 0,$$

$$(6) \quad \delta p = V_s^2\delta\varrho,$$

(for notations see [1]).

The divergence condition (3) gives:

$$(7) \quad h_3 = 0.$$

*) For relevant literature see [1].

If we consider wave solutions of Eqs. (1)-(6), we have for amplitudes

(8) $i\sigma u_1^* - (4\pi\varrho)^{-1} H_3 ik h_1^* = 0$,

(9) $i\sigma u_2^* - (4\pi\varrho)^{-1} H_3 ik h_2^* - 2u_3^* \Omega = 0$,

(10) $i\sigma u_3^* + (4\pi\varrho)^{-1} H_2 ik h_2^* + 2u_2^* \Omega + V_s^2 \varrho^{-1} ik\, \delta\varrho^* - ik\, \delta\psi^* = 0$,

(11) $i\sigma h_1^* - ikH_3 u_1^* + \nu_m k^2 h_1^* = 0$,

(12) $i\sigma h_2^* + ikH_2 u_3^* - iH_3 k u_2^* + \nu_m k^2 h_2^* = 0$,

(13) $i\sigma\, \delta\varrho^* + ik\, \varrho u_3^* = 0$,

(14) $-k^2\, \delta\psi^* + 4\pi G\, \delta\varrho^* = 0$,

where σ denotes the frequency, k — the wave number, and asterisks — the corresponding amplitudes.

From (13), (14) and (8) we obtain

$$\delta\varrho^* = -k\sigma^{-1} \varrho u_3^* , \tag{15}$$

$$\delta\psi^* = 4\pi G\, \delta\varrho^* k^{-2} = -4\pi G k^{-1} \sigma^{-1} \varrho u_3^* , \tag{16}$$

$$u_1^* = (4\pi\varrho)^{-1} H_3 k \sigma^{-1} h_1^* . \tag{17}$$

Substituting (15)-(17) into (10), (11) and introducing the following symbols

$$\Omega_J = (V_s^2 k^2 - 4\pi G\varrho)^{1/2} , \tag{18}$$

$$\Omega_A = kH_3 (4\pi\varrho)^{-1/2} , \tag{19}$$

$$\Omega_B = kH_2 (4\pi\varrho)^{-1/2} , \tag{20}$$

$$\Omega_R = k^2 \nu_m , \tag{21}$$

we obtain the system of equations in the form

$$\sigma u_2^* + i2\Omega u_3^* - (4\pi\varrho)^{-1/2} \Omega_A h_2^* = 0 , \tag{22}$$

$$-i2\Omega\sigma u_2^* + (\sigma^2 - \Omega_J^2) u_3^* + \sigma (4\pi\varrho)^{-1/2} \Omega_B h_2^* = 0 , \tag{23}$$

$$(\sigma^2 - i\Omega_R \sigma - \Omega_A^2) h_1^* = 0 , \tag{24}$$

$$(4\pi\varrho)^{1/2} \Omega_A u_2^* - (4\pi\varrho)^{1/2} \Omega_B u_3^* + (i\Omega_R - \sigma) h_2^* = 0 . \tag{25}$$

Dispersion equation and condition for instability

The system of Eqs. (22)-(25) has nontrivial solutions, if the following dispersion equation is satisfied

$$\begin{vmatrix} \sigma \;, & 2\Omega i \;, & 0 \;, & -(4\pi\varrho)^{-1/2} \Omega_A \\ -i2\Omega\sigma \;, & \sigma^2 - \Omega_J^2 \;, & 0 \;, & \sigma (4\pi\varrho)^{-1/2} \Omega_B \\ 0 \;, & 0 \;, & \sigma^2 - i\Omega_R \sigma - \Omega_A^2 \;, & 0 \\ (4\pi\varrho)^{1/2} \Omega_A \;, & -(4\pi\varrho)^{1/2} \Omega_B \;, & 0 \;, & -(\sigma - i\Omega_R) \end{vmatrix} = 0. \tag{26}$$

This equation can be written as follows

$$(27)\qquad (\sigma^2 - i\Omega_R\sigma - \Omega_A^2)\cdot\begin{vmatrix} \sigma\ , & -i2\Omega\ , & \Omega_A \\ i2\Omega\sigma\ , & \sigma^2 - \Omega_J^2\ , & \sigma\Omega_B \\ \Omega_A\ , & \Omega_B\ , & \sigma - i\Omega_R \end{vmatrix} = 0\,,$$

so that

$$(28)\qquad (\sigma^2 - i\Omega_R\sigma - \Omega_A^2)\{\sigma^4 - i\sigma^3\Omega_R - \sigma^2(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2) + \\ + i\sigma\Omega_R(\Omega_J^2 + 4\Omega^2) + \Omega_A^2\Omega_J^2\} = 0\,.$$

The above equation gives the instability condition and is equivalent to the alternative of the following conditions

$$(29)\qquad \sigma^2 - i\Omega_R\sigma - \Omega_A^2 = 0\,,$$

$$(30)\qquad \sigma^4 - i\sigma^3\Omega_R - \sigma^2(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2) + i\sigma\Omega_R(\Omega_J^2 + 4\Omega^2) + \Omega_A^2\Omega_J^2 = 0\,.$$

From (29) we obtain two modes of wave propagation

$$(31)\qquad \sigma_{(1),(2)} = i\tfrac{1}{2}\Omega_R \pm \tfrac{1}{2}(4\Omega_A^2 - \Omega_R^2)^{1/2}\,.$$

In the case of

$$(32)\qquad \operatorname{sign}(4\Omega_A^2 - \Omega_R^2) = +1\,,$$

we obtain periodic oscillations quenched by the factor

$$(33)\qquad \exp(-\tfrac{1}{2}\Omega_R)t\,.$$

In the case of

$$(34)\qquad \operatorname{sign}(4\Omega_A^2 - \Omega_R^2) = -1\,,$$

the solutions (31) give the quenching of the type

$$(35)\qquad \exp - \{\tfrac{1}{2}\Omega_R \pm [(\tfrac{1}{2}\Omega_R)^2 - \Omega_A^2]^{1/2}\}t\,.$$

In the case of infinite electrical conductivity the two solutions (31) reduce to the two Alfven waves

$$(36)\qquad \sigma_{(1),(2)} = \pm\Omega_A\,,$$

running in opposite directions.

When

$$(37)\qquad \Omega_A = 0\,,$$

the solutions (31) give

$$(38)\qquad \sigma_{(1)} = 0\,,$$

$$(39)\qquad \sigma_{(2)} = i\Omega_R\,.$$

Thus, all solutions given by (29) cannot in any case give the instability of the considered medium.

Now we consider solutions given by Eq. (30). After the substitution

$$\omega = i\sigma \,, \tag{40}$$

Eq. (30) takes the form

$$W(\omega) = \omega^4 + \omega^3 \Omega_R + \omega^2(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2) + \\ + \omega\Omega_R(\Omega_J^2 + 4\Omega^2) + \Omega_A^2\Omega_J^2 = 0 \,. \tag{41}$$

If the parallel component of the magnetic field exists

$$\Omega_A \neq 0 \,, \tag{42}$$

Jeans's instability criterion remains unaffected by the finite electrical conductivity of the medium.

This theorem can be proved as follows.

If

$$\operatorname{sign} \Omega_J^2 = -1 \,, \tag{43}$$

so that

$$\operatorname{sign} W(0) = -1 \tag{44}$$

and because of

$$\lim_{\omega \to +\infty} W(\omega) = +\infty \,, \tag{45}$$

at least one real positive root exists. Therefore in case (43) the medium is unstable.

If

$$\operatorname{sign} \Omega_J^2 = +1 \,, \tag{46}$$

$W(\omega)$ as a polynom with positive coefficients cannot have a positive real root. If $W(\omega)$ has complex roots, $\operatorname{Re}\omega$ satisfies the following equation

$$\begin{aligned} Y(\operatorname{Re}\omega) = {} & 64(\operatorname{Re}\omega)^6 + 96\Omega_R(\operatorname{Re}\omega)^5 + \\ & + \{48\Omega_R^2 + 32(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2)\}(\operatorname{Re}\omega)^4 + \\ & + \{8\Omega_R^3 + 32\Omega_R(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2)\}(\operatorname{Re}\omega)^3 + \\ & + \{4(\Omega_J^2 - \Omega_A^2)^2 + 4(\Omega_B^2 + 4\Omega^2)^2 + 8(\Omega_J^2 + \Omega_A^2)(\Omega_B^2 + 4\Omega^2) + \\ & + 8\Omega_R^2(\Omega_J^2 + \Omega_A^2 + \Omega_B^2 + 4\Omega^2) + 4\Omega_R^2(\Omega_J^2 + 4\Omega^2)\}(\operatorname{Re}\omega)^2 + \\ & + 2\Omega_R\{(\Omega_J^2 - \Omega_A^2)^2 + 2(\Omega_J^2 + \Omega_A^2)(\Omega_B^2 + 4\Omega^2) + \\ & + (\Omega_B^2 + 4\Omega^2)^2 + \Omega_R^2(\Omega_J^2 + 4\Omega^2)\}\operatorname{Re}\omega + \\ & + \Omega_R^2\{\Omega_J^2\Omega_B^2 + 4\Omega^2(\Omega_A^2 + \Omega_B^2)\} = 0 \,, \end{aligned} \tag{47}$$

all coefficients of which are positive; thus $\operatorname{Re}\omega$ cannot have a positive and the medium is stable.

As we see in (32), Jeans's criterion is unaffected by the effect of finite electrical conductivity.

Now we consider the case, in which the parallel component of the magnetic field vanishes

(48) $$\Omega_A = 0\,.$$

In this case Eq. (41) reduces to the following one

(49) $$X(\omega) = \omega^3 + \omega^2\Omega_R + \omega(\Omega_J^2 + \Omega_B^2 + 4\Omega^2) + \Omega_R(\Omega_J^2 + 4\Omega^2) = 0\,.$$

If

(50) $$\operatorname{sign}(\Omega_J^2 + 4\Omega^2) = -1\,,$$

so that

(51) $$\operatorname{sign} X(0) = -1\,,$$

and because of

(52) $$\lim_{\omega\to+\infty} X(\omega) = +\infty$$

one at least real positive root exists.

In the case of

(53) $$\operatorname{sign}(\Omega_J^2 + 4\Omega^2) = +1\,,$$

the polynom $X(\omega)$, all coefficients of which are positive, cannot have any positive real root. Real parts of the complex roots of the polynom $X(\omega)$ satisfy the following equation with positive coefficients

(54) $$Z(\operatorname{Re}\omega_{(2),(3)}) = 8(\operatorname{Re}\omega)^3 + 8\Omega_R(\operatorname{Re}\omega)^2 + \\ + 2(\Omega_R^2 + \Omega_B^2 + 4\Omega^2 + \Omega_J^2)\operatorname{Re}\omega + \Omega_R\Omega_B^2 = 0\,,$$

so that

(55) $$\operatorname{sign}(\operatorname{Re}\omega_{(2),(3)}) = -1\,,$$

and the instability cannot take place in the considered medium.

Therefore in the case (48) the effect of finite conductivity of the medium removes the influence of the magnetic field on the critical instability wavelength.

Hence, the following theorem is proved:

The criterion for gravitational instability of the rotating medium with finite electrical conductivity remains unaffected by the magnetic field, independently from the existence of the component of the magnetic field, parallel to the direction of wave propagation.

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THEORETICAL PHYSICS

Composite Variational Problems

by

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Presented by W. RUBINOWICZ on September 1, 1959

In modern mathematics variational problems are usually discussed using methods of functional analysis in abstract spaces (as Banach and Hilbert spaces). Some investigators occasionally mention composite operators (e. g. [1] p. 61). As far as the author knows, however, an explicite presentation of methods of solution of composite variational problems is lacking in the literature on the subject. In this paper we shall give, therefore, a short discussion of such a problem met in some application to physics [2].

In analogy to composite functions (functions of functions) of the type $F(x, G(x)) = H(x)$ it may be considered a composite functional

$$U[f] = V[f, W[f]] = \int_{t_1}^{t_2} \mathcal{L}\{f(t), \dot{f}(t), \ldots, \overset{(n)}{f}(t), t, W[f]\}\, dt\,, \tag{1}$$

where

$$W[f] = \int_{t_1}^{t_2} M\{f(t'), \dot{f}(t'), \ldots, \overset{(m)}{f}(t'), t'\}\, dt'\,. \tag{2}$$

We are interested in finding a stationary function of U, i. e. such a function $f(t)$ that

$$\delta U[f] = 0\,. \tag{3}$$

Substituting in (1), as usually, $f(t)$ by $f(t) + ag(t)$, where a is a parameter and $g(t)$ an arbitrary function vanishing for $t = t_1$ and $t = t_2$, we get from (3)

$$\left(\frac{dU}{da}\right)_{a=0} = \int_{t_1}^{t_2} \left\{\frac{\partial \mathcal{L}(t)}{\partial f} g(t) + \frac{\partial \mathcal{L}(t)}{\partial \dot{f}} \dot{g}(t) + \ldots + \frac{\partial \mathcal{L}(t)}{\partial \overset{(n)}{f}} \overset{(n)}{g}(t)\right\} dt + \tag{4}$$

$$+ \int_{t_1}^{t_2} dt \frac{\partial \mathcal{L}(t)}{\partial W} \int_{t_1}^{t_2} \left\{\frac{\partial M(t')}{\partial f} g(t') + \frac{\partial M(t')}{\partial \dot{f}} \dot{g}(t') + \ldots + \frac{\partial M(t')}{\partial \overset{(m)}{f}} \overset{(m)}{g}(t')\right\} dt' = 0\,.$$

Changing the notation of integration variables of the double integral in (4) (t for t' and reversely) and integrating both integrals by parts a sufficient number of times, we obtain

$$(5) \qquad \left(\frac{dU}{da}\right)_{a=0} = \int_{t_1}^{t_2} \left\{\frac{\partial \mathcal{L}(t)}{\partial f} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}(t)}{\partial \dot{f}}\right) + \ldots + (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial \mathcal{L}(t)}{\partial \overset{(n)}{f}}\right) + \right.$$

$$\left. + \lambda \left[\frac{\partial M(t)}{\partial f} - \frac{d}{dt}\left(\frac{\partial M(t)}{\partial \dot{f}}\right) + \ldots + (-1)^m \frac{d^m}{dt^m}\left(\frac{\partial M(t)}{\partial \overset{(m)}{f}}\right)\right\} g(t)\, dt = 0\,,$$

where

$$(6) \qquad \lambda = \int_{t_1}^{t_2} \frac{\partial \mathcal{L}(t')}{\partial W} dt' = \frac{\partial V[f, W[f]]}{\partial W} = \mu[f, W[f]] = \lambda[f]\,.$$

Assuming that all occurring functions are sufficiently regular we may apply to (5) Du Bois-Reymond's theorem and we get an equation of the Euler-Ostrogradski type

$$(7) \qquad \frac{\partial L}{\partial f} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{f}}\right) + \ldots + (-1)^k \frac{d^k}{dt^k}\left(\frac{\partial L}{\partial \overset{(k)}{f}}\right) = 0\,.$$

Here, k denotes the greater of numbers n and m, and

$$(8) \qquad L = L(W, \lambda) = \mathcal{L}(W) + \lambda M\,.$$

Eq. (7) is not purely differential but integro-differential because of the dependence of L on functionals $W[f]$ and $\lambda[f]$. Functionals W and λ are, however, independent of t and we may use, therefore, a method analogous to that of variation of constants in the theory of differential equations. By means of such a method we solve (7) in two steps. First, we consider W and λ as constants independent of f and solve (7) as a purely differential equation with parameters W and λ. We get

$$(9) \qquad f = f(t, W, \lambda)\,,$$

when some initial conditions for f are given. Secondly, we insert (9) into (2) and (6) and carrying out integration over t, we obtain two finite (algebraic or transcendental) equations for values of constants W and λ

$$(10) \qquad \begin{cases} W = P(W, \lambda)\,, \\ \lambda = Q(W, \lambda)\,. \end{cases}$$

Solving the system of Eqs. (10) with respect to W and λ and substituting the results into (9) we get the final solution $f(t)$ of our problem.

Basic Eq. (7) may be easily written by means of the notation of variational derivatives. We have

$$(11) \qquad \frac{\partial U}{\partial f} \equiv \left(\frac{\delta V[f, W[f]]}{\delta f}\right)_{W=\mathrm{const.}} + \left(\frac{\partial V[f, W[f]]}{\partial W}\right)_{f=\mathrm{conv.}} \frac{\delta W}{\delta f} = 0$$

in exact analogy to the formula for a derivative of a composite function $F(x, G(x)) \equiv H(x)$ (equal to zero when an extremum of H is sought)

$$(12) \qquad \frac{dH}{dz} \equiv \left(\frac{\partial F(x, G(x))}{\partial x}\right)_{G=\text{const.}} + \left(\frac{\partial F(x, G(x))}{\partial G}\right)_{x=\text{const.}} \frac{dG}{dz} = 0 .$$

It is obvious that (11) may be also obtained by abstract methods of functional analysis on account of formal properties of variational derivatives or, more generally, of gradients of operators (cf. [1] p. 61).

The problem discussed may be generalized directly and solved as above for an arbitrary (but finite) number of functions $f_1, \ldots, f_p$, of independent variables $t_1, \ldots, t_q$, and of functionals $W_1, \ldots, W_r$.

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THEORETICAL PHYSICS

A Fixed-Source Approach to Scattering of Kaons, Supplement

by
W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on September 4, 1959

In paper [1] on a fixed-source approach to scattering of kaons the separation of scattering amplitudes for different isospins T was carried out incorrectly (there are some errors in formulae ($\widetilde{16}$), (20), ($\widetilde{20}$) and ($\widetilde{21}$)). In the present note this point of argument is corrected.

The scattering amplitudes describing the processes $N+K \to N+K$ and $N+\bar{K} \to N+\bar{K}$ respectively, can be written down in terms of partial amplitudes as follows

$$(1)\quad T_{\sigma\vec{k}a}(\sigma'\vec{k}'a') = -4\pi \frac{v(k')v(k)}{\sqrt{4\omega_{k'}\omega_k}} \sum_T P_T(\sigma'a', \sigma a)\Big[g_T(\omega_k) + \sum_J P_J(\sigma'\vec{k}', \sigma\vec{k}) h_{JT}(\omega_k)\Big]$$

and

$$(\tilde{1})\quad T_{\sigma\vec{k}\tilde{a}}(\sigma'\vec{k}'\tilde{a}') = -4\pi \frac{v(k')v(k)}{\sqrt{4\omega_{k'}\omega_k}} \sum_T \widetilde{P}_T(\sigma'a', \sigma a)\Big[\widetilde{g}_T(\omega_k) + \sum_J P_J(\sigma'\vec{k}', \sigma\vec{k}) \widetilde{h}_{JT}(\omega_k)\Big],$$

where

$$(2)\quad P_{\frac{1}{2}}(\sigma'\vec{k}', \sigma\vec{k}) = [(\vec{\sigma}\cdot\vec{k}')(\vec{\sigma}\cdot\vec{k})]_{\sigma'\sigma}, \qquad P_{\frac{3}{2}}(\sigma'\vec{k}', \sigma\vec{k}) = [3\vec{k}'\cdot\vec{k} - (\vec{\sigma}\cdot\vec{k}')(\vec{\sigma}\cdot\vec{k})]_{\sigma'\sigma},$$

$$(3)\quad P_0(\sigma'a', \sigma a) = \tfrac{1}{4}(\delta_{\sigma'\sigma}\delta_{a'a} - \vec{\tau}_{\sigma'\sigma}\cdot\vec{\tau}_{a'a}), \qquad P_1(\sigma'a', \sigma a) = \tfrac{1}{4}(3\delta_{\sigma'\sigma}\delta_{a'a} + \vec{\tau}_{\sigma'\sigma}\cdot\vec{\tau}_{a'a})$$

and

$$(4)\qquad \widetilde{P}_T(\sigma'a', \sigma a) = (\tau_2)_{a'\beta'} P_T(\sigma'\beta', \sigma\beta)(\tau_2)_{a\beta}.$$

For further explanation of the notation used compare [1]. The projection operators P_J's and P_T's, $\widetilde{P}_T$'s, satisfy the following relations

$$(5)\quad \sum_J P_{J'}(\sigma'\vec{k}', \sigma\vec{k}) A_{J'J} = P_J(\sigma'\vec{k}, \sigma\vec{k}'), \qquad A = (A_{J'J}) = \tfrac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}$$

and

$$(6)\quad \sum_{T'} P_{T'}(\sigma'\alpha', \sigma\alpha) B_{T'T} = \widetilde{P}_T(\sigma'\alpha, \sigma\alpha'), \qquad B = (B_{T'T}) = \tfrac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}.$$

We may write the following dispersion relations, if we carry out in (12) and $(\widetilde{12})$ of [1] the separation of partial amplitudes:

$$(7)\quad \begin{aligned} g_T(\omega) &= g_T^B(\omega) + \frac{1}{(2\pi)^2}\int\limits_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')}\left[\frac{\sigma_T^S(\omega')}{\omega'-(\omega+i\epsilon)} + \frac{\sum\limits_{T'} B_{TT'}\widetilde{\sigma}_{T'}^S(\omega')}{\omega'+\omega}\right], \\ h_{JT}(\omega) &= h_{JT}^B(\omega) + \frac{1}{(2\pi)^2}\int\limits_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{1}{k'v^2(k')}\left[\frac{\sigma_{JT}^P(\omega')}{\omega'-(\omega+i\epsilon)} + \frac{\sum\limits_{J'T'} A_{JJ'}B_{TT'}\widetilde{\sigma}_{J'T'}^P(\omega')}{\omega'+\omega}\right] \end{aligned}$$

and

$$(\widetilde{7})\quad \begin{aligned} \widetilde{g}_T(\omega) &= \widetilde{g}_T^B(\omega) + \frac{1}{(2\pi)^2}\int\limits_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')}\left[\frac{\widetilde{\sigma}_T^S(\omega')}{\omega'-(\omega+i\epsilon)} + \frac{\sum\limits_{T'} B_{TT'}\sigma_{T'}^S(\omega')}{\omega'+\omega}\right], \\ \widetilde{h}_{JT}(\omega) &= \widetilde{h}_{JT}^B(\omega) + \frac{1}{(2\pi)^2}\int\limits_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{1}{k'v^2(k')}\left[\frac{\widetilde{\sigma}_{JT}^P(\omega')}{\omega'-(\omega+i\epsilon)} + \frac{\sum\limits_{J'T'} A_{JJ'}B_{TT'}\sigma_{J'T'}^P(\omega')}{\omega'+\omega}\right]. \end{aligned}$$

The Born terms have here, thanks to (15) and $(\widetilde{15})$ of [1], the forms

$$(8)\quad \begin{cases} g_T^B(\omega) = -\dfrac{\lambda}{4\pi} \quad (T = 0, 1), \\ h_{J0}^B(\omega) = \dfrac{1}{3}\begin{Bmatrix} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{Bmatrix} \dfrac{1}{4\pi\mu_K^2}\dfrac{1}{2}\left(-\dfrac{f_{N\Lambda}^2}{\Delta M_\Lambda+\omega} + \dfrac{3f_{N\Sigma}^2}{\Delta M_\Sigma+\omega}\right), \\ h_{J1}^B(\omega) = \dfrac{1}{3}\begin{Bmatrix} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{Bmatrix} \dfrac{1}{4\pi\mu_K^2}\dfrac{1}{2}\left(\dfrac{f_{N\Lambda}^2}{\Delta_\Lambda M+\omega} + \dfrac{f_{N\Sigma}^2}{\Delta M_\Sigma+\omega}\right) \end{cases}$$

and

$$(\widetilde{8})\quad \begin{cases} \widetilde{g}_T^B(\omega) = -\dfrac{\lambda}{4\pi} \quad (T = 0, 1), \\ \widetilde{h}_{J0}^B(\omega) = \begin{Bmatrix} 1 & J = \frac{1}{2} \\ 0 & J = \frac{3}{2} \end{Bmatrix} \dfrac{1}{4\pi\mu_K^2}\dfrac{f_{N\Lambda}^2}{\Delta M_\Lambda-\omega}, \\ \widetilde{h}_{J1}^B(\omega) = \begin{Bmatrix} 1 & J = \frac{1}{2} \\ 0 & J = \frac{3}{2} \end{Bmatrix} \dfrac{1}{4\pi\mu_K^2}\dfrac{f_{N\Sigma}^2}{\Delta M_\Sigma-\omega}. \end{cases}$$

If we use the approximation $\Delta M_\Lambda = \Delta M_\Sigma$ $(= \Delta M)$, we get

$$(9)\quad h_{JT}^B(\omega) = \frac{\lambda_{JT}}{\Delta M+\omega}; \qquad \lambda_{JT} = \frac{f_T^2}{4\pi\mu_K^2}\frac{1}{3}\begin{Bmatrix} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{Bmatrix},$$

$$f_T^2 = \tfrac{1}{2}\begin{Bmatrix} -f_{N\Lambda}^2 + 3f_{N\Sigma}^2 & T = 0 \\ f_{N\Lambda}^2 + f_{N\Sigma}^2 & T = 1 \end{Bmatrix}$$

and

$$(\widetilde{9})\quad \widetilde{h}^{B}_{JT}(\omega)=\frac{\widetilde{\lambda}_{JT}}{\Delta M-\omega},\quad \widetilde{\lambda}_{JT}=\frac{\widetilde{f}^{2}_{T}}{4\pi\mu_K^2}\begin{Bmatrix}1 & J=\frac{1}{2}\\ 0 & J=\frac{3}{2}\end{Bmatrix},\quad \widetilde{f}^{2}_{T}=\begin{Bmatrix}f^2_{N\Lambda} & T=0\\ f^2_{N\Sigma} & T=1\end{Bmatrix}.$$

If one supposes that the charge exchange scattering of K and $\bar{K}$ mesons is small in the present theory (it is suggested by the Born approximation), then $g_0\approx g_1$, $h_{J0}\approx h_{J1}$ and $\widetilde{g}_0\approx\widetilde{g}_1$, $\widetilde{h}_{J0}\approx\widetilde{h}_{J1}$, and hence $\sigma_0^S\approx\sigma_1^S$, $\sigma_{J0}^P\approx\sigma_{1J}^P$ and $\widetilde{\sigma}_0^S\approx\widetilde{\sigma}_1^S$, $\widetilde{\sigma}_{J0}^P\approx\widetilde{\sigma}_{J1}^P$. In such a case the dispersion relations (7) and $(\widetilde{7})$ take the forms (20) and $(\widetilde{20})$ of [1], respectively.

Applying to the P waves the effective range approximation we obtain (in the approximation $\Delta M_\Lambda=\Delta M_\Sigma$)

$$k^3 v(k)\,h^B_{JT}(\omega^*)\cot\delta^P_{JT}(\omega^*)=1-r^P_{JT}(\Delta M+\omega^*)+\dots,$$

where $\omega^*=k^2/2M_N+\omega=k^2/2M_N+\sqrt{k^2+\mu_K^2}$ and

$$r^P_{JT}=\frac{1}{\lambda_{JT}}\frac{1}{(2\pi)^2}\int\limits_{\Delta M+\mu_\pi}^{\infty}d\omega'\frac{1}{k'\omega' v^2(k')}\Big[\sigma^P_{JT}(\omega')+\sum_{J'T'}A_{JJ'}R_{TT'}\widetilde{\sigma}^P_{J'T'}(\omega')\Big].$$

Hence,

$$r^P_{\frac{3}{2}1}=\frac{3}{2}\frac{4\pi\mu_K^2}{f_1^2}\frac{1}{(2\pi)^2}\int\limits_{\Delta M+\mu_\pi}^{\infty}d\omega'\frac{1}{k'\omega' v^2(k')}\big(\sigma^P_{\frac{3}{2}1}+\tfrac{1}{3}\widetilde{\sigma}^P_{\frac{1}{2}0}+\tfrac{1}{3}\widetilde{\sigma}^P_{\frac{1}{2}1}+\tfrac{1}{6}\widetilde{\sigma}^P_{\frac{3}{2}0}+\tfrac{1}{6}\widetilde{\sigma}^P_{\frac{3}{2}1}\big).$$

We can see that $r^P_{\frac{3}{2}1}>0$, giving evidence of a resonance in the state $P_{\frac{3}{2}}T=1$ in the scattering $N+K\to N+K$. If the charge exchange scattering of K mesons is small, then also $r^P_{\frac{3}{2}0}>0$ and we have evidence of a $P_{\frac{3}{2}}T=0$ resonance in the scattering $N+K\to N+K$.

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THEORETICAL PHYSICS

Effective Range Approximation for Scattering of Kaons with Scalar Coupling

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on September 25, 1959

Recently, Białkowski and Jurewicz carried out Tamm-Dancoff calculations including recoils for the reaction $p+K^+ \to p+K^+$ in the case of pseudoscalar Yukawa coupling [1]. They got a $P_{3/2}$ resonance at high energies, which shifts to higher and higher energies with decreasing coupling constant. The effective range approximation applied to fixed-source dispersion relations also suggests such a $P_{3/2}\,T=1$ resonance in the pseudoscalar case [2]. In the present note the fixed-source approach and the effective range approximation are used to discuss the scattering of kaons in the scalar case.

We consider the interaction Hamiltonian of kaons with the fixed-source baryon in the form

$$H^{BK} = f^0 \xi_\alpha \int d_3 x\, \varrho(|\vec{x}|)\left(CK(\vec{x})\right)_\alpha + \text{h. c.} \tag{1}$$

For explanation of the notation used compare [2] and [3].

The transition amplitudes for the reactions $N+K \to N+K$ and $N+\bar{K} \to N+\bar{K}$ respectively, can be represented as follows

$$T_{\sigma\vec{k}\alpha}(\sigma'\vec{k}'\alpha') = -4\pi \frac{v(k')\,v(k)}{\sqrt{4\omega_{k'}\omega_k}} \sum_T P_T(\sigma'\alpha', \sigma\alpha)\, g_T(\omega_k) \tag{2}$$

and

$$T_{\sigma\vec{k}\tilde{\alpha}}(\sigma'\vec{k}'\tilde{\alpha}') = -4\pi \frac{v(k')\,v(k)}{\sqrt{4\omega_{k'}\omega_k}} \sum_T \tilde{P}_T(\sigma'\alpha', \sigma\alpha)\, \tilde{g}_T(\omega_k), \tag{$\tilde{2}$}$$

where g_T and $\tilde{g}_T$ $(T=0,1)$ describe the S waves, P_T and $\tilde{P}_T$ $(T=0,1)$ are the isospin projection operators given in [2]. For the functions g_T and $\tilde{g}_T$ we get the following dispersion relations

$$g_T(\omega) = g_T^B(\omega) + \frac{1}{(2\pi)^2} \int\limits_{\Delta M_\Lambda + \mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')} \left[\frac{\sigma_T^S(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\sum\limits_{T'} B_{TT'} \tilde{\sigma}_{T'}^S(\omega')}{\omega' + \omega}\right] \tag{3}$$

and

$$(\tilde{3}) \qquad \tilde{g}_T(\omega) = \tilde{g}_T^B(\omega) + \frac{1}{(2\pi)^2} \int\limits_{\Delta M_\Lambda + \mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')} \left[\frac{\tilde{\sigma}_T^S(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\sum\limits_{T'} B_{TT'} \sigma_T^S(\omega')}{\omega' + \omega} \right],$$

where

$$(4) \qquad B = (B_{TT'}) = \tfrac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

and the Born terms have the forms

$$(5) \qquad \begin{cases} g_0^B(\omega) = \dfrac{1}{4\pi} \dfrac{1}{2} \left(-\dfrac{f_{N\Lambda}^2}{\Delta M_\Lambda + \omega} + \dfrac{3 f_{N\Sigma}^2}{\Delta M_\Sigma + \omega} \right), \\[2ex] g_1^B(\omega) = \dfrac{1}{4\pi} \dfrac{1}{2} \left(\dfrac{f_{N\Lambda}^2}{\Delta M_\Lambda + \omega} + \dfrac{f_{N\Sigma}^2}{\Delta M_\Sigma + \omega} \right) \end{cases}$$

and

$$(\tilde{5}) \qquad \begin{aligned} \tilde{g}_0^B(\omega) &= \frac{1}{4\pi} \frac{f_{N\Lambda}^2}{\Delta M_\Lambda - \omega}, \\ \tilde{g}_1^B(\omega) &= \frac{1}{4\pi} \frac{f_{N\Sigma}^2}{\Delta M_\Sigma - \omega} \end{aligned}$$

$(\Delta M_\Lambda = M_\Lambda - M_N\,,\ \Delta M_\Sigma = M_\Sigma - M_N)$.

Applying to dispersion relations (3) and $(\tilde{3})$ the effective range approximation of the Chew-Low type, we obtain in the case of real phase shifts δ_T^S $(T = 0, 1)$:

$$(6) \qquad k v(k) g_T^B(\omega^*) \cot \delta_T^S(\omega^*) = \mathrm{Re} \frac{g_T^B(\omega^*)}{g_T(\omega^*)} = 1 - r_T^S(\Delta M + \omega^*) + \dots,$$

where $\omega^* = k^2/2M_N + \sqrt{k^2 + \mu_K^2}$ is the total energy in the centre-of-mass system and

$$(7) \qquad r_T^S = -\frac{d}{d\omega} \frac{g_r^B(\omega)}{g_T(\omega)} \Bigg|_{\Delta M + \omega = 0}$$

$$= \frac{4\pi}{f_T^2} \frac{1}{(2\pi)^2} \int\limits_{\Delta M + \mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')\,\omega'} \left[\sigma_T^S(\omega') + \sum_T B_{TT'} \tilde{\sigma}_{T'}^S(\omega') \right].$$

We used above the approximation $M_\Lambda = M_\Sigma$ and denoted $\Delta M = \Delta M_\Lambda = \Delta M_\Sigma$,

$$(8) \qquad \begin{aligned} f_0^2 &= \tfrac{1}{2}(-f_{N\Lambda}^2 + 3 f_{N\Sigma}^2)\,, \\ f_1^2 &= \tfrac{1}{2}(f_{N\Lambda}^2 + f_{N\Sigma}^2)\,. \end{aligned}$$

It follows from (7) that

$$r_1^S = \frac{4\pi}{f_1^2}\frac{1}{(2\pi)^2}\int\limits_{\Delta M+\mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')\,\omega'}\left(\sigma_1^S + \frac{1}{2}\tilde{\sigma}_0^S + \frac{1}{2}\tilde{\sigma}_1^S\right). \tag{9}$$

We can see that $r_1^S > 0$, and this is evidene of a resonance in the state $S_{1/2}\,T = 1$ in the scattering $N+K \to N+K$. It seems that the effective range r_1^S is much greater than the $r_{\frac{3}{2}1}^P$ obtained in the pseudoscalar case [2], because of the experimentally known predominance of S waves in the scattering of K^+ mesons at low energies. Thus, the $S_{1/2}\,T = 1$ resonance in the scalar case should lie at lower energies than the $P_{3/2}\,T = 1$ resonance in the pseudoscalar case. On the other hand, it is well known that no resonance in the scattering $p+K^+ \to p+K^+$ has so far been observed. Our results and the results of paper [1] seem, therefore, to suggest that the Yukawa coupling $\bar{N}YK$ is pseudoscalar and that there exists for kaons also another strong coupling, maybe, the Barshay coupling $\bar{K}K\pi\pi$.

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THEORETICAL PHYSICS

On a Generalization of Peierls' Theorem

by

J. CZERWONKO

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In our previous note [1] we have considered the generalization of the well-known variational method of Bogoljubov. The method developped in [1] presents good results in the following three cases:

(i) for the classical systems,

(ii) for the representation in which the contribution to the partition function coming from the non-diagonal part of the Hamiltonian is small,

(iii) for the representation of eigenfunctions of the energy operator.

In other cases we must generalize the theorem given by Peierls [2] (see also [1]).

Using Dirac's notation we may write the operator $e^{-\beta H}$ in the form

$$\sum_E |E\rangle\, e^{-\beta E} \langle E|\,,$$

where $|E\rangle$ form a complete orthonormal set and $H|E\rangle = E|E\rangle$.

The diagonal element of the operator $e^{-\beta H}$ in another representation of orthonormal vectors, $|n\rangle$ is given by:

$$\sum_E |\langle n|E\rangle|^2 e^{-\beta E}\,. \tag{1}$$

It follows from [1] that, if $p_E \geqslant 0$, $\dfrac{d^{2l}\varphi}{dE^{2l}} \geqslant 0$, the following inequality holds

$$\sum_E p_E\varphi(E) \geqslant \varphi(M)\sum_E p_E + \tfrac{1}{2}\varphi''(M)\sum_E p_E(E-M)^2 + \\ + \dots + \frac{1}{(2l-1)!}\varphi^{(2l-1)}(M)\sum_E p_E(E-M)^{2l-1}\,, \tag{2}$$

where

$$M = \Big(\sum_E p_E E\Big)\Big(\sum_E p_E\Big)^{-1}.$$

If we choose $p_E = |\langle E|n\rangle|^2$ and $\varphi(E) = e^{-\beta E}$ we obtain

$$(3) \qquad \langle n|e^{-\beta H}|n\rangle \geqslant e^{-\beta\langle n|H|n\rangle}\left[1+\sum_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(\langle n|H|n\rangle - H)^i|n\rangle\right],$$

since $\sum\limits_E p_E = 1$ and $M = \sum\limits_E Ep_E = \langle n|H|n\rangle = H_n$. From (3) we obtain

$$(4) \qquad Sp\, e^{-\beta H} \geqslant \sum_n e^{-\beta H_n}\left[1+\sum_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(H_n-H)^i|n\rangle\right]$$

which is the generalization of a theorem given by Peierls.

If the sum

$$\sum_n e^{-\beta H_n}\langle n|(H_n-H)^i|n\rangle$$

cannot be elementary calculated, we may employ, to estimate the right-hand side of (4), the generalized method of Bogoljubov given in [1]. It follows immediately from (2) that for $v_n \geqslant 0$

$$\sum_n v_n e^{-\beta H_n'} \geqslant e^{-\beta S}\left\{\sum_n v_n\left[1+\sum_{r=2}^{2m-1}\frac{\beta^r}{r!}(S-H_n')^r\right]\right\},$$

where $S = \left(\sum\limits_n v_n H_n'\right)\left(\sum\limits_n v_n\right)^{-1}$.

If we choose $v_n = e^{-\beta H_{0n}}\left(1+\sum\limits_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(H_n-H)^i|n\rangle\right)$, and $H_n = H_n' + H_{0n}$ we obtain from (4)

$$(5) \qquad Spe^{-\beta H} \geqslant e^{-\beta S}\left\{\sum_n e^{-\beta H_{0n}}\left[1+\sum_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(H_n-H)^i|n\rangle\right]\times\right.$$
$$\left.\times\left[1+\sum_{r=2}^{2m-1}\frac{\beta^r}{r!}(S-H_n')^r\right]\right\}.$$

In this case two possibilities exist to introduce the variational parameters: (i) using the representation $|n\rangle$ which depends on some parameters (see e. g. Kubo [3]) and (ii) dividing the H_n into H_{0n} and H_n'.

Similarly as in [1] we may estimate the value $Spe^{-\beta H}$ by majorizing it. We have

$$(6) \qquad Spe^{-\beta H} \leqslant \sum_n e^{-\beta H_n}\left[1+\sum_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(H_n-H)^i|n\rangle\right]+$$
$$+\frac{\beta^{2l}}{(2l)!}e^{-\beta \operatorname{Min}_E E}\sum_n\langle n|(H_n-H)^{2l}|n\rangle.$$

Similarly, applying the method of [1] to the right-hand side of (6), we obtain

$$(7)\qquad Sp\, e^{-\beta H} \leqslant e^{-\beta S}\Big\{\sum_{n} e^{-\beta H_{0n}}\Big[1+\sum_{i=2}^{2l-1}\frac{\beta^i}{i!}\langle n|(H_n-H)^i|n\rangle\Big]\times$$

$$\times\Big[1+\sum_{r=2}^{2m-1}\frac{\beta^r}{r!}(S-H'_n)^r\Big]\Big\}+\frac{\beta^{2l}}{(2l)!}e^{-\beta \operatorname{Min}_{E} E}\sum_{n}\langle n|(H_n-H)^{2l}|n\rangle$$

$$+\frac{\beta^{2m}}{(2m)!}e^{-\beta \operatorname{Min}_{n} H_n}\sum_{n} v_n(S-H'_n)^{em}\,.$$

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

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В. МЛЯК, О ЛИНЕЙНОМ ДИФФЕРЕНЦИАЛЬНОМ НЕРАВЕНСТВЕ ПАРАБОЛИЧЕСКОГО ТИПА . . . . стр. 653–656

В работе приводятся положения, касающиеся регулярности функции u, при которых доказывается, что из неравенства $\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{dx^2}$, имеющего место почти везде, вытекает неравенство $u \leqslant 0$.

В доказательстве использованы некоторые свойства функции Грина для первой краевой задачи уравнения теплопроводности.

Х. БОХЕНЕК, АНАЛИТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЯ ЭЙКОНАЛА . . . . стр. 657–660

Результаты работы автора применимы к исследованию асимптотических разложений решений уравнения Гельмгольца. Дается теорема и ее доказательство об аналитических свойствах эйконала, построенного как огибающая семейства плоскостей.

Рассматривается область D, расположенная вне некоторой выпуклой кривой C. Выбираем начало системы полярных координат внутри кривой C; пусть ее уравнение $r = f(\vartheta)$. Определим в области D функцию с параметром α

(1) $$l(x, \alpha) = \boldsymbol{v}(\alpha)\cdot \boldsymbol{x} + b(\alpha),$$

где вектор $\boldsymbol{v}$ имеет полярные координаты 1 и α, а $\boldsymbol{x}$ является позиционным вектором. Обозначим через $\Pi(\alpha)$ полупрямую, определенную следующим образом:

(2) $$\boldsymbol{v}'\cdot \boldsymbol{x} + b' = 0 \qquad |\vartheta - \alpha| < \frac{\pi}{2}.$$

Справедлива следующая теорема: *если функция $b(\alpha)$ является аналитической, регулярной на действительной оси, принимает на этой оси действительные значения и ее период 2π и если, кроме того, для каждого α выполнены следующие неравенства*:

(3) $$-f\left(\alpha + \frac{\pi}{2}\right) < b'(\alpha) < f\left(\alpha - \frac{\pi}{2}\right),$$

а также

(4) $$v(a)\cdot x_c - b''(a) \geqslant \lambda_0 > 0\,,$$

где x_c является точкой пересечения полупрямой $\Pi(a)$ с кривой C, а λ_0 постоянная, то при этих условиях уравнение и условие (2) определяют в области D однозначную функцию $a_0(x)$.

Эта функция и есть аналитической функцией координат. Подставляя в формуле (1) a_0 вместо a, получаем однозначную и аналитическую в области D функцию $L(x) = l(x, a_0)$, удовлетворяющую уравнению эйконала

(5) $$(\nabla L)^2 = 1\,.$$

Ю. МУСЕЛЯК и В. ОРЛИЧ, НЕКОТОРЫЕ ЗАМЕЧАНИЯ О МОДУЛЯРНЫХ ПРОСТРАНСТВАХ . . . . стр. 661–668

Функционал $-\infty < \varrho(x) \leqslant +\infty$, определенный на линейном пространстве X, назовем модуляром, если он удовлетворяет условиям: А.1, А.2 и А.3. Пространство $\bar{X}_0^* = \{x \in X : a_n \to 0$ имплицирует $\varrho(a_n x) \to 0\}$ назовем модулярным пространством: при норме $\|x\| = \inf\{\varepsilon > 0 : (x/\varepsilon) \leqslant \varepsilon\}$ оно является пространством типа F^*. Кроме сходимости по норме вводится в $\bar{X}_\varrho^*$ более слабое понятие сходимости, а именно модулярная сходимость. Последовательность $\{x_n\} \in \bar{X}_0^*$ назовем модулярно сходящейся к $x_0 \in \bar{X}_0^*$, если существует $k > 0$ такое что $\varrho[k(x_n - x_0)] \to 0$. Если $\bar{X}_0^*$ структура, а модуляр $\varrho(x)$ удовлетворяет условиям (1)—(8), то $\bar{X}_0^*$ назовем модулярной структурой.

Целью настоящей работы является обсуждение вопроса существования нетривиальных линейных функционалов, непрерывных по отношению к модулярной сходимости. Оказывается (см. Теорема 6.1) что, если $\bar{X}_0^*$ безатомная модулярная структура и если $\lim\limits_{k\to\infty} \varrho(kx)/k = 0$ в некотором множестве $x > 0$ модулярно плотном в положительном конусе пространства $\bar{X}_0^*$, то на $\bar{X}_\varrho^*$ имеется лишь тривиальный линейный функционал, модулярно непрерывный.

Заметим, что выражение „безатомность” обозначает, что для каждого положительного $x \in \bar{X}^*$ существуют положительные элементы $y, z \in \bar{X}_\varrho^*$ такие, что $x = y + z$ и $y \wedge z = 0$. В частности, если μ—конечная и безатомная мера на σ-алгебре подмножеств множества E, а $M(u)$ является четной и непрерывной функцией, возрастающей для $u > 0$, $M(0) = 0$ и если $\varrho(x) = \int\limits_E M[x(t)]\,d\mu$, то на пространстве $\bar{X}_\varrho^* = L_M^*$ имеются нетривиальные линейные функционалы, модулярно непрерывные тогда и только тогда, когда $\lim\limits_{u\to\infty} M(u)/u > 0$ (см. теорема 6.31).

Кроме того, в работе доказано (см. Теорема 4.11), что если $M(u)$, сверх того, выпукла, а $N(u)$ обозначает функцию дополнительную к $M(u)$ в смысле Юнга, то общим видом линейного модулярно непрерывного функционала на L_M^* является следующее выражение $\xi(x) = \int\limits_E x(t)y(t)\,d\mu$, где $y \in L_N^*$.

Пусть L обозначает класс функций, для которых существует трансформата Лапласа.

Теорема. *Если* $f(t) \in L$ *и* $\int\limits_0^t (t-\tau) e^{\tau^2} d\tau \in L$, *то* $f(t) = 0$ *почти везде в интервале* $0 \leqslant t < \infty$.

Теорема верна лишь при предположении $f(t) \in L$.

В работе дается положительный ответ на следующий вопрос: можно ли построить поле вероятности в декартовом произведении $(R_n * R_n)$ двух пространств конечномерных (R_n) при помощи нормализированного решения $\big(U(X, t, Y, \tau),\ X \in R_n,\ Y \in R_n,\ 0 \leqslant t < \tau\big)$ первого уравнения А. Н. Колмогорова с коэффициентами Гельдера.

Приводится метод исследования направлений вертексов в случае когда располагаем только данными о собственных движениях малочисленных звездных групп. Разработанный автором метод применен в качестве примера к проверке предположения латышского астронома Икаунинекса [1], состоящего в том, что вертекс углеродных звезд перпендикулярен к галактической плоскости. Использованы при этом данные, касающиеся звезд типа N и R, содержащиеся в каталоге GC. Полученные результаты не подтверждают предположения Икаунинекса.

Показано, что условие гравитационной неустойчивости вращающейся среды с конечной (скалярной) электропроводностью не изменяется при наличии магнитного поля независимо от несуществования компоненты магнитного поля, параллельной направлению распространения волны.

В работе сформулирована сложная вариационная проблема для одной неизвестной функции одной независимой переменной и одного функционала, от которого зависит исследуемый функционал. Показано, что решение этой проблемы может быть сведено к решению одного дифференциального уравне-

ния типа Эйлера-Остроградского и системы двух конечных уравнений (алгебраических или трансцендентных) для двух неизвестных постоянных, от которых параметрически зависит дифференциальное уравнение. Этот результат может быть непосредственно обобщен на более общие сложные вариационные проблемы.

В работе [1] сегрегация амплитуд рассеяния для разных изотропных спинов была проведена не вполне аккуратно. Настоящая работа исправляет эту неточность.

В работе применяется приближение эффективного радиуса для описания рассеяния мезонов K на жестком источнике при скалярном сопряжении Юкавы.

Получается возможность резонанса в состоянии $S_{1/2}\,T=1$, тогда как в псевдоскалярном случае эта возможность получается для состояния $P_{1/2}\,T=1$ (как это было показано в предыдущей работе автора).

В заметке обобщено известное неравенство Пейерльса путем введения в него недиагональной части оператора энергии. Это обобщенное неравенство мы применяем для усовершенствования полученного нами в предыдущей заметке обобщенного вариационного метода исчисления статистических сумм, являющегося обобщением известного вариационного метода Н. Н. Боголюбова.